HOMEWORK #3 SOLUTIONS

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1. FCOT

C.6.1. Notice that

$$\begin{aligned}\langle x+y, x+y\rangle - \langle x-y, x-y\rangle &= (\langle x,x\rangle + 2\langle x,y\rangle + \langle y,y\rangle) - (\langle x,x\rangle - 2\langle x,y\rangle + \langle y,y\rangle \\ &= 4\langle x,y\rangle.\end{aligned}$$

Dividing both sides by four and noting that $\langle x \pm y, x \pm y\rangle = \|x \pm y\|^2$ yeilds the desired result.

C.6.3. Note: if you showed this problem for $\mathbb{R}^2$ instead of $\mathbb{R}^n$ you lost half a point.

We will show that the only norm on $\mathbb{R}^n$ generated by an inner product is $\|\cdot\|_2$. Suppose that $\|\cdot\|_p$ is generated by an inner product; this implies that the parallelogram law holds. Let $x = (1, 0, \ldots, 0)$ and $y = (0, 1, 0, \ldots, 0)$. $\|x + y\|_p = \|x - y\|_p = 2^{1/p}$. But $\|x\|^2 = \|y\|^2 = 1$. Thus we get

$$4 = 2^{1/p+1}$$

which holds only when $p = 2$. Therefore we know that the only p-norm that could be generated by an inner product is $\|\cdot\|_2$, which is generated by the standard inner product, so the only p-norm generated by an inner product is the 2-norm, as desired.

C.6.4. We will show that if a sequence $\{x_n\}$ converges to x under d_1 it converges to x under d_2, and if X is open under d_1 then it is open under d_2. Since the arguments will be symmetric in d_1 and d_2, this will show the desired results.

a. Suppose that $\{x_n\}$ converges to x under d_1. Then we know that for $\epsilon > 0$ there exists N such that $d_1(x_n, x) < \epsilon/c_1$ for $n > N$. Notice, however, that this implies that $d_2(x_n, x) \leq c_1 d_1(x_n, x) < \epsilon$, so for any ϵ we can find an N such that $d_2(x_n, x) < \epsilon$ for $n > N$. Thus the sequence $\{x_n\}$ converges to x under d_2, as desired.

b. Now suppose that X is open under d_1. This means that X^c is closed under d_1; we will show that it is also closed under d_2. Let $\{x_n\}$ be a convergent sequence in X^c under d_1. It converges to a point $x \in X^c$. However, by part (a), we know that it also converges to x under d_2. Thus we know that X^c contains all of its limit points under d_2. so it is closed. Therefore, X is open.

C.6.6. All of the p-metrics imposed on $\mathbb{R}^2$ are equivalent. By transitivity of equivalence of metrics it suffices to show that $\|\cdot\|_p$ is equivalent to $\|\cdot\|_1$.

Let $S = \{x \in \mathbb{R}^n : \|x\| = 1$. This is closed and bounded, so it is compact. Let $f(x) = (x_1^p + \cdots + x_n^p)^{1/p} = \|x\|_p$. We know that this function is continuous. Then let $m = \min_S f(x)$ and $M = \max_S f(x)$; note that neither of these is 0. Then $\|x\|_1 \leq \frac{1}{m}\|x\|_p$, since $\|(x/\|x\|_1)\|_p \geq m$ (by linearity of norm and definition of m). Simiarly, $\|x\|_p \leq M\|x\|_1$. Thus we see that these are equivalent, as desired.

C.6.8. First notice that the discrete metric is positive definite by definition. In addition, since if $x \neq y$ then $y \neq x$ we know that $d(x, y) = d(y, x) = 1$ when $x \neq y$, (and symmetry is trivial when $x = y$) so it is symmetric. Now consider the triangle inequality. Suppose that $x \neq y$. Then either $x \neq z$ or $y \neq z$, so $1 = d(x, y) \leq d(x, z) + d(z, y)$, since at least one of the quantities on the RHS must be nonzero. If $x = y$ then the triangle inequality holds because the metric is positive definite. Thus the discrete metric is actually a metric.

C.6.9. Let $X \subset \mathbb{R}$. Consider any point $x \in X$. $B(x, 1/2) = \{x\}$, so x is an interior point and X is open. But by analogous reasoning, we see that X^c is open, so X is closed. Thus any subset of $\mathbb{R}$ under the discrete metric is closed.

Now consider a subset X, and suppose that it is compact. Notice that it is not enough to note that X is closed and bounded, since under this metric $\mathbb{R}$ is not a normed linear space. Notice that if X is infinite we can take a sequence where every term is distinct and get a sequence that never converges. Thus no infinite set is compact. However, we know that all finite sets are compact, so we see that the compact subsets of $\mathbb{R}$ under the discrete metric are exactly the finite ones.

2. OVSM

2.16.12. a. Suppose that $\mathbf{x} = (\xi_1, \xi_2, \ldots) \in \ell_{p_0}$, and let $p \geq p_0$. Since $\sum |\xi_i|^{p_0}$ is finite, we know that $\lim_{i \to 0} \xi_i = 0$. Thus there is an N such that for all $n > N$, $\xi_n < 1$. Then

$$\sum_{n=1}^{\infty} |\xi_n|^p = \sum_{n=1}^{N} |\xi_n|^p + \sum_{n=N+1}^{\infty} |\xi_n|^p \leq \sum_{n=1}^{N} |\xi_n|^p + \sum_{n=N+1}^{\infty} |\xi_n|^{p_0},$$

where the last step is because $x^{p_0} \geq x^p$ if $0 \leq x < 1$. Now notice that each of the two sums on the right-hand side of the above are finite, since the first is a finite sum and the right is the tail of a convergent series. Thus we know that $\mathbf{x} \in \ell_p$.

b. Let N be the same as in part (a). Now notice that

$$\lim_{p \to \infty} \sum_{n=N+1}^{\infty} |\xi_n|^p = 0.$$

This follows because for any ϵ we can pick an M such that the sum of the tail past M is less than $\epsilon/2$, and then we can choose p such that $|\xi_n|^p < \epsilon/2M$ for all $N < n \leq M$, so we can find a p such that the sum above is less than ϵ.

Let $\xi = \sup_n |\xi_n|$. It suffices to show that $\lim_{p \to \infty} \left\| \frac{\mathbf{x}}{\xi} \right\|_p = 1$. From now on we will assume that $\xi = 1$; this is sufficient for the problem because norms are consistent with scalar multiplication. Notice that

$$\left(\sum_{n=1}^{\infty} |\xi_n|^p \right)^{1/p} \geq (|\xi_n|^p)^{1/p} = |\xi_n|$$

for all n, so in particular we know that $\lim_{p\to\infty} \|\mathbf{x}\|_p \geq |\xi_n|$ for all n, so $\lim_{p\to\infty} \|\mathbf{x}\|_p \geq 1$. But

$$\begin{aligned}
\lim_{p\to\infty} \sum_{n=1}^{\infty} |\xi_n|^p &= \lim_{p\to\infty} \left(\sum_{n=1}^{N} |\xi_n|^p + \sum_{n=N+1}^{\infty} |\xi_n|^p \right) \\
&= \lim_{p\to\infty} \sum_{n=1}^{N} |\xi_n|^p + \lim_{p\to\infty} \sum_{n=N+1}^{\infty} |\xi_n|^p \\
&\leq \lim_{p\to\infty} \sum_{n=1}^{N} 1^p + \lim_{p\to\infty} \sum_{n=N+1}^{\infty} |\xi_n|^p \\
&= 1.
\end{aligned}$$

Since we have $1 \leq \lim_{p\to\infty} \|\mathbf{x}\|_p \leq 1$ we have shown the desired result.

2.16.14. Let $\{v_1, \ldots, v_n\}$ be a basis for V, and let $\{e_1, \ldots, e_n\}$ be the standard basis for E^n. Define $T : V \to E^n$ by $\sum_{i=1}^{n} a_i v_i \mapsto \sum_{i=1}^{n} a_i e_i$. We claim that this map is an isomorphism.

First notice that T is linear by definition. It is surjective because we can write any vector $v \in E^n$ as $\sum_{i=1}^{n} a_i e_i$, and then the vector $\sum_{i=1}^{n} a_i v_i \in V$ maps to it. T is injective because for a sum $\sum_{i=1}^{n} a_i e_i = 0$ we must have $a_i = 0$ for all i, and therefore that the reimage was also zero. Thus we have shown that T is an isomorphism.

2.16.15. a. First note that the sum of two sequences is another sequence, and that multiplying a sequence by a constant produces another sequence. In addition, since distributive laws are defined termwise, they follow from the definition of the original vector space.
Thus we see that Y, the space of all sequences, is a linear space. We now need to show that X, the subset consisting of all Cauchy sequences, is a subspace. For this it suffices to show that the sum of any two Cauchy sequences is Cauchy, and that multiplication by a constant leaves a Cauchy sequence Cauchy.
Let $x = \{x_n\}, y = \{y_m\}$ be two Cauchy sequences. We know that for each ϵ there exist N, M such that $d(x_a, x_b) < \epsilon/2$ for all $a, b > N$ and $d(y_a, y_b) < \epsilon/2$ for all $a, b > M$. This means that $d(x_a + y_a, x_b + y_b) \leq d(x_a, x_b) + d(y_a, y_b) < \epsilon$ for all $a, b > \max(M, N)$, so $\{x_n + y_n\}$ is Cauchy. Similarly, for any fixed α for all ϵ there exists N such that $d(x_a, x_b) < \epsilon/\alpha$ for all $a, b > N$, so $d(\alpha x_a, \alpha x_b) < \epsilon$ for these a and αx is also a Cauchy sequence. Thus X is a subspace, so it is a linear space.

b. Suppose that $\|x\| = 0$. Then we know that $\sup_n |x_n| = 0$, so $x_n = 0$ for all n, and therefore $x = (0, 0, \ldots)$. Similarly, if $x \neq (0, 0, \ldots)$ there exists an m such that $x_m \neq 0$, so $\sup_n |x_n| \geq |x_m| > 0$. So this function is positive definite.
Also, notice that $\|\alpha x\| = \sup_n |\alpha x_n| = |alpha| \sup_n |x_n|$ so the norm is linear.
Now $\|x + y\| = \sup_n |x_n + y_n| \leq \sup_n(|x_n| + |y_n|) = \sup_n |x_n| + \sup_n |y_n| = \|x\| + \|y\|$. So this actually defines a norm.

c. First, notice that M is a closed subspace. Suppose that we have a point $m \in M^c$, so $m = \{m_i\}$ with $m_i \not\to 0$. Notice that there must exist an N, $\epsilon > 0$ such that for $n > N$, $\|m_n\| > \epsilon$ (otherwise the sequence must converge to 0). Notice that

$B(m, \epsilon/2) \subset M^c$, since all terms in each sequence (past a certain point) have norm that is bounded below by $\epsilon/2$. Thus M is closed, so $\bar{X}/M$ is complete.
Consider the subspace $Y \subset \bar{X}$ consisting of the constant sequences. We claim that this is dense. Let $\{x_n\} \in \bar{X}$. We claim that for any $\epsilon > 0$ we can find a $y \in Y$ such that $d(\{x_n\}, y) < \epsilon$. We know that there exists an N such that for $m, n > N$, $d(x_n, x_m) < \epsilon/2$. Define $y = \{x_{N+1}\} \in Y$. Notice that

$$d(\{x_n\}, y) = \inf_{\{m_i\} \in M} \sup_n |x_n + m_n - x_{N+1}| \leq \sup_{n > N} |x_n - x_{N+1}| \leq \epsilon/2 < \epsilon,$$

where the third step is because we can take $\{m_i\} = (x_1, x_2, \ldots, x_N, 0, 0, \ldots)$. Thus Y is dense.
It remains to show that there is an isometric isomorphism $X \to Y$. But this is clear, just by using the map $x \mapsto \{x\}$, which is clearly both an isometry and an isomorphism. So we are done.