Math 116

Some Solutions for Assignment II

1.7.5) For $n \geq 1$, take

$$x_n = \begin{cases} 0, & n \text{ odd} \\ n, & n \text{ even.} \end{cases}$$

Clearly any subset of the indices which has a finite number of even indices is a convergent subsequence and converges to zero. Also, any subset of the indices which has an infinite number of even indices cannot converge, which also implies that the series itself does not converge.

1.7.6) Using the triangle inequality that all metrics must satisfy, we know that

$$d(x, y) \leq d(x, x_k) + d(x_k, y_k) + d(y_k, y)$$

In the same way,

$$d(x_k, y_k) \leq d(x_k, x) + d(x, y) + d(y, y_k)$$

Now take any $\epsilon > 0$. Since the sequence $\{x_k\}$ converges to x, there is an N_x such that $d(x_k, x) < \frac{\epsilon}{2}$ for any $k > N_x$. Similarly, there is an N_y such that $d(y_k, y) < \frac{\epsilon}{2}$ for any $k > N_y$. Let $N = \max(N_x, N_y)$. Then for any $k > N$, the above two equations give

$$\begin{aligned} d(x, y) &< d(x_k, y_k) + \epsilon \\ d(x_k, y_k) &< d(x, y) + \epsilon \end{aligned}$$

which we can write

$$d(x, y) < d(x_k, y_k) + \epsilon < d(x, y) + 2\epsilon$$

Subtracting an ϵ from each, this says

$$d(x, y) - \epsilon < d(x_k, y_k) < d(x, y) + \epsilon$$

or

$$|d(x, y) - d(x_k, y_k)| < \epsilon$$

and this holds for all $k > N$. Since $\epsilon > 0$ was arbitrary, this says exactly that the sequence $d(x_k, y_k)$ converges to $d(x, y)$.

1.7.27)
1. Not compact, since $A \cup B = [-1, 0) \cup (0, 1]$, which isn't closed, and hence not compact.
2. Not compact, since $A + B = (-1, 1)$, which also isn't closed.
3. Compact, since $A \cap B = \emptyset$ which is trivially compact.

1.7.28) We claim that

$$\overline{A} = \bigcap_{\substack{S \supset A \\ S \text{ closed}}} S$$

is the desired set. Clearly $A \subset \overline{A}$, since every set S in the intersection contains A. $\overline{A}$ is closed because an arbitrary intersection of closed sets is closed. If C is any other closed set containing A, $\overline{A} \subset C$ because C intersected with anything else is always contained in C. Thus $\overline{A}$ is the set we want. Note: you can also define $\overline{A} = A \cup \partial A$, where ∂A is the set of limit points of A. The proof proceeds simlarly.

1.7.45) f continuous implies that for all $\epsilon > 0$ there exists a $\delta > 0$ s.t. $\|x - p\| < \delta$ implies $|f(x) - f(p)| < \epsilon$. Let $\epsilon_{f(p)} = f(p)$. Then there is a $\delta_{f(p)}$ s.t. $\|x - p\| < \delta_{f(p)}$ implies $|f(x) - f(p)| < f(p)$. So $B = B(p; \delta_{f(p)})$, the open ball around p of size $\delta_{f(p)}$, and you're done.

1.7.46) We first note that the set $\{x \mid f(x) = 0\}$ is usually denoted $\text{Ker}f$. Thus we want to show $\text{Ker}f$ closed in X. To do so, note that $\text{Ker}f = f^{-1}(\{0\})$. But $\{0\}$ is closed in $\mathbb{R}$, so the inverse image of $\{0\}$ is closed in X (this is just the definition of f continuous).

3.4.15) First, π continuous on $\mathbb{R}$ implies that $\pi + \pi + \pi$ is continuous on $\mathbb{R}^3$. Next, $[0, y_1]$ is closed and bounded, which in $\mathbb{R}$ means it's compact. Since f is continuous, Bolzano-Weierstrass tells us that f achieves its maximum M on $[0, y_1]$, so thus $y_2 = f(y_1 - x_1)$ is bounded by M. Thus $[0, y_2]$ is also closed and bounded, hence (in $\mathbb{R}$) compact, and the same reasoning shows $[0, y_3]$ is compact.

12.7.1) Let the limit of the subsequence $\{x_{n_i}\}$ be x; we claim that the original sequence converges to x as well. Take any $\epsilon > 0$. Since the sequence is Cauchy, there is some M_1 such that $|x_i - x_j| < \frac{\epsilon}{2}$ whenever $i, j > M_1$. Also, since the subsequence converges, there is some M_2 such that $|x - x_{k_i}| < \frac{\epsilon}{2}$ whenever $k_i > M_2$. Let $M = \max(M_1, M_2)$. For $j, k_i > M$, we have

$$\begin{aligned} |x - x_j| &= |x - x_{k_i} + x_{k_i} - x_j| \\ &\leq |x - x_{k_i}| + |x_{k_i} - x_j| \\ &\leq \frac{\epsilon}{2} + \frac{\epsilon}{2} \\ &\leq \epsilon \end{aligned}$$

Hence the sequence $\{x_j\}$ converges to x, as we hoped to show.

12.7.2) **(a)** Yes, $[0,1]$ is a closed subset of $\mathbb{R}$, and a closed subset of a complete space is itself complete.

(b) No, lots of examples why not. Try the power series for e, or decimal expansion for $\sqrt{2}$.

(c) No, try e/n as $n \to \infty$.

(d) Yes, since any Cauchy sequence in $\mathbb{N}$ must be constant for all n greater than some N (due to the rigid spacing of 1 between each member of $\mathbb{N}$). Thus the series is convergent, and $\mathbb{N}$ is complete.

(e) No, try $1/n$ as $n \to \infty$.

(f) Yes, since S, the reciprocals of the integers with zero, is a closed subset of $[0,1]$. Proof: Fix any point x in the complement of S. Then $x \neq 0$ and $x \neq 1$. Also, suppose $x > 0$. Then there exists an integer n s.t. $1/n < x$, since $1/n \to 0$ as $n \to \infty$. So x must lie between two successive reciprocals, i.e. there is some n such that

$$\frac{1}{n+1} < x < \frac{1}{n}$$

Now we just take ϵ to be the minimum distance of the distance from x to either $\frac{1}{n}$ or $\frac{1}{n+1}$, the ball of radius ϵ around x is in the complement of S. Hence the complement of S is open in $[0,1]$, and so S is closed. Since S is a closed subset of a complete space, it is complete.

(g) No, try $2 - 1/n$ as $n \to \infty$.

12.7.8) First, we show pointwise convergence. Fix $x \in [0,1]$. If $x = 0$ or $x = 1$, then $f_n(x) = x$ for all n, and we're done. Suppose $x \in (0,1)$. Then $f_n(x) = x^n$, so clearly $f_n(x) \to 0$ as $n \to \infty$. This shows pointwise convergence. Now, the functions $f_n(x)$ are all bounded and continuous. If f_n converged uniformly to f, then the Uniform Convergence Theorem (12.9 in Sundaram) would tell us that f is continuous. However, the function f is not continuous, and so we know that the sequence f_n does not converge uniformly to f.