HOMEWORK #1 SOLUTIONS

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1. FCOT

1.7.2 The additional condition that we need to impose is that $\inf x_n > 0$. We will show that this condition is both necessary and sufficient.

First suppose that $\inf x_n = \epsilon > 0$, and $x = 0$. Then we know that there exists a subsequence $\{x_{n_i}\}_{i=0}^{\infty}$ of $\{x_n\}$ such that $\lim_{i\to\infty} x_{n_i} = 0$. However, this implies that there exists N such that for all $n > N$, $d(x_{n_i}, x) < \epsilon/2$ and in particular that $x_{n_{N+1}} < \epsilon/2$, which contradictics the fact that $x_n > \epsilon$ for all n. Thus $\inf x_n > 0$ implies $x > 0$, so it is sufficient.

Now suppose that $\inf x_n = 0$. We will show that this implies that 0 is a limit point of x_n. Let $y_0 = 0$ and $y_i = \min\{n \in \mathbb{Z}^+ : x_n < 1/i,\ n > y_{i-1}\}$ for $i > 0$. We claim that y_i will be well defined for all i. If we can show this then we will know that $\{x_{y_i}\}$ is a subsequence of $\{x_n\}$ which converges to 0. Suppose that for some j there is an N such that for all $n > N$, $x_n > 1/j$ (which is what would be required for some y_i to not exist). But this means that $\inf x_n = \min(\inf_{1\le n\le N} x_n, \inf_{n>N} x_n) \ge \min(\inf_{1\le n\le N} x_n, 1/j) > 0$ because each x_i is positive. However, this is a contradiction, since we assumed that $\inf x_n = 0$, so we are done.

1.7.16 On this problem .5 points were taken away if there was no justification given for most of the cases.

a. Notice that we know that the irrational numbers are dense. That means that there is an irrational number x_n in each range $[1 - 1/n, 1]$, so we know that $\sup X = 1$. Analogously, we know that $\inf X = 0$. Since neither 0 nor 1 are in X we know that X does not have a minimum or a maximum.

b. Since $1/n \le 1$ for all n and equality holds at $n = 1$, we have $\sup X = \max X = 1$. For each ϵ we can find n such that $1/n < \epsilon$, so we know that $\inf X = 0$. Since $0 \notin X$, we know that X does not have a minimum.

c. Analogously to part (b) we see that $\inf X = \min X = 0$ and $\sup X = 1$, but X has no maximum.

d. Notice that for $\sin x > 1/2$ we need $\pi/6 < x < 5\pi/6$, so we have $X = (\pi/6, 5\pi/6)$. Thus we know that $\inf X = \pi/6$, $\sup X = 5\pi/6$, but X has neither a minimum or a maximum.

1.7.17 Consider $[0,1]^c = (-\infty, 0) \cup (1, \infty)$. We want to show this is open. Let $y \in [0,1]^c$. If $y < 0$ then $B(y, |y|) \subset [0,1]^c$, and if $y > 1$ $B(y, y-1) \subset [0,1]^c$, so $[0,1]^c$ is open, and therefore $[0,1]$ is closed.

Consider $(0,1)$. If $y \in (0,1)$ then $B(y, \min(y, (1-y)/2)) \subset (0,1)$ so $(0,1)$ is open.

Now consider $[0,1)$; we will show that this is neither open nor closed. Notice that $-\epsilon/2 \in B(0,\epsilon)$ but is not in $[0,1)$, so $[0,1)$ is not open. Also, the sequence $\{1 - 1/n\}$ is in $[0,1)$ but it does not contain its limit, 1, so we know that $[0,1)$ is not closed.

The argument for $(0,1]$ is completely analogous to the above argument, with the roles of 0 and 1 reversed.

1.7.26 Notice that $B((1.5,1.5),\epsilon)$ contains the point $(1.5, 1.5+\epsilon/2)$, which is not in A, so A is not open. The sequence $\{1+1/n, 1+1/n\}$ is in A but its limit point, $(1,1)$, is not, so A is not closed. Also, if $(x,y)\in A$ then $\sqrt{x^2+y^2}<\sqrt{4+4}=2\sqrt{2}$, so A is bounded. However, since A is not closed it is not compact.

1.7.27 a. $A\cup B=[-1,1]-\{0\}$, in which the sequence $\{1/n\}$ does not have a convergent subsequence. So it is not compact.

b. $A+B=(-1,1)$, which is not closed, so it is not compact.

c. $A\cap B=\emptyset$ which is compact.

1.7.32 Notice that A is a subset of $\mathbb{R}$ that is bounded (it is contained in $B(0,2)$, for example). So it is closed if and only if it is compact.

Notice that

$$A^c=(-\infty,0)\cup\left(\bigcup_{n=1}^{\infty}\left(\frac{1}{n+1},\frac{1}{n}\right)\right)\cup(1,\infty)$$

which is a union of open sets. Thus it is open, so A is closed, and therefore also compact.

(This could also be shown by proving that 0 is the only limit point of A.)

2.6.3 In this problem, justification was not necessary to get full credit.

Part	$f(\mathcal{D})$	$\inf f(\mathcal{D})$	$\sup f(\mathcal{D})$	$\min f(\mathcal{D})$	$\max f(\mathcal{D})$
(a)	$[1,2]$	1	2	1	2
(b)	$[1,2]$	1	2	1	2
(c)	$[0,1)\cup\{2\}$	0	2	0	2
(d)	$[0,3)$	0	3	0	N/A

2.6.4 If f is increasing it is not enough for $f(\mathcal{D})$ to be compact. For a counterexample, consider the function in part (c) of the previous problem.

3.4.5 Since $\lim_{x\to\infty}f(x)=0$ there exists an N such that for $x>N$ $f(x)<1$. Then notice that $f(0)>f(x)$ for all $x>N$, so f cannot attain a global maximum on $[N,\infty)$. Thus f has a global maximum if and only if it has one on $[0,N]$. However, this is a compact set, so f attains a maximum on this set, and therefore a global maximum.

However, the same is not necessarily true for a minimum. Consider the function $1/(x+1)$. Its infimum is 0, but it never attains it.

3.4.8 Let $U=\{x\in\mathbb{R}^n_+ : u(x)\geq u\}$. We want to minimize $p\cdot x$ for $x\in U$. Notice that U is the preimage of $[u,\infty)$, so it is closed. Let $u_0\in U$ be any vector, and let $V=\{x\in\mathbb{R}^n_+ : p\cdot x\leq p\cdot u_0\}$. Notice that V is the preimage of $[0,p\cdot u_0]$ under the continuous function $x\mapsto p\cdot x$, so it is also closed. Also notice that the desired minimum is in the set $V\cap U$.

Now notice that $V\cap U$ is closed, since it is the intersection of two closed sets. If we can show that it is bounded we will have that it is compact, and therefore contains the desired minimum. We will show that V is bounded, which suffices to show the desired statement. Notice that $p\cdot x\leq p\cdot u_0$ implies that $p_ix_i\leq p\cdot u_0$, so $x_i\leq(p\cdot u_0)/p_i$, which implies that $\|x\|\leq(p\cdot u_0)\sqrt{\frac{1}{p_1^2}+\cdots+\frac{1}{p_n^2}}$, which is independent of x, so V is bounded, and we are done.

Notice that if $p \not\gg 0$ we have the problem that $p \cdot x$ can become negative, and therefore V can be unbounded on its intersection with U, so a minimum of $p \cdot x$ might not exist. If u is not continuous U might not be closed, so we might not be able to say anything about the existence of a minimum.