Solutions to Homework #4, Math 116

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Problem 3.1 (Luenberger)

Let x, y be vectors in a pre-Hilbert space. Show that $|\langle x|y\rangle| = \|x\| \cdot \|y\|$ iff $\exists x, y \in \mathbb{R}$ s.t. $\alpha x + \beta y = 0$.

Solution to 3.1

This is really just Cauchy-Schwartz, which says that $|\langle x|y\rangle|^2 \leq \|x\|^2\|y\|^2 = \langle x|x\rangle \langle y|y\rangle$, with equality iff x and y are linearly dependent. This is done in lemma 1 on page 47 of Luenberger. $\square$

Problem 3.3 (Luenberger)

Let H consist of all $m \times n$ real matrices with addition and scalar multiplication defined as the usual corresponding operations with matrices and when the inner product of two matrices is defined as

$$(A \mid B) = \text{Trace}\left[A'QB\right]$$

where A' denotes the transpose of the matrix A and Q is sa symmetric, positive-definite $m \times m$ matrix. Prove that H is a Hilbert space.

Solution to 3.3

First we need to show that $(A \mid B) = \text{Trace}\,[A'QB]$ defines an inner product. Then we'll show that H with respect to the norm $\|X\| = \sqrt{(X \mid X)}$ is complete.

Inner products must satisfy four properties:

1. $(A \mid B) = \overline{(B \mid A)}$

   As H is defined to be the space of real valued matrices, our base field is $\mathbb{R}$ and thus $\overline{(B \mid A)} = (B \mid A)$.

$$\begin{aligned}(A \mid B) &= \text{Trace}\left[A'QB\right] \\ &= \text{Trace}\left[(B'QA)'\right] \\ &= \text{Trace}\left[B'QA\right] \\ &= (B \mid A)\end{aligned}$$

   By the facts that $\text{Trace}\,[A] = \text{Trace}\,[A']$ and that $(AB)' = B'A'$.

2. $(A + B \mid C) = (A \mid C) + (B \mid C)$

$$\begin{aligned}(A + B \mid C) &= \text{Trace}\left[(A+B)'QC\right] \\ &= \text{Trace}\left[A'QC + B'QC\right] \\ &= \text{Trace}\left[A'QC\right] + \text{Trace}\left[B'QC\right] \\ &= (A \mid C) + (B \mid C)\end{aligned}$$

3. $(\lambda A \mid B) = \lambda(A \mid B)$

$$\begin{aligned}
(\lambda A \mid B) &= \text{Trace}\left[(\lambda A)' Q B\right] \\
&= \text{Trace}\left[\lambda A' Q B\right] \\
&= \lambda \text{Trace}\left[A' Q B\right] \\
&= \lambda(A \mid B)
\end{aligned}$$

4. $(A \mid A) \geq 0$ and $(A \mid A) = 0 \iff A = \mathbf{0}$

   Since Q is positive definite, we know that for every $x \in R^m$, $x \neq 0$, $x'Qx > 0$. If we write $A = [x_1, x_2, \ldots, x_n]$, where x_i is the *ith* column of A, then $A'QA$ has entries of the form $(x_i)'Qx_j$. Since Q is positive definite, every entry of $A'QA$ is greater than or equal to zero. Thus, Trace$[A'QA] \geq 0$. Equality only happens which all the columns of A are equal to 0.

This shows H is a pre-Hilbert space. Now we need to show that every Cauchy sequence in H converges to an element of H. Let $\{A_n\}$ be a Cauchy sequence in H. This means that for $\epsilon > 0, \exists N \text{such that} n, m > N \rightarrow \|A_n - A_m\| < \epsilon$. $\|A_n - A_m\| = \sqrt{\text{Trace}\left[(A_n - A_m)'Q(A_n - A_m)\right]}$ This is a linear combination of elements of $A_n - A_m$. Since each entry of the diagonal of $(A_n - A_m)'Q(A_n - A_m)$ is positive, $\{A_n\}$ cauchy implies that the sequence $\{(a_{ij}^{(n)} - a_{ij}^{(m)})\}$ is Cauchy in R. Since R is complete, each of these sequences converges to a real number. We let A be the matrix $\{A_n\}$ converges to point-wise. Clearly, A is an $m \times n$ real-valued matrix.

Now we need to show $\{A_n\} \rightarrow A$ in the inner-product norm. Trace$\left[(A_n - A)'Q(A_n - A)\right]$ depends only on a linear combination of elements of the form $(a_{ij}^{(n)} - a_{ij})$. There are at most $n(n-1)/2$ such elements (as Q is symmetric). We need each element to be less than some fraction of ϵ. For each entry, there exists an N_{ij} such that this holds for $n > N_{ij}$. We let $N = \max_{i,j} N_{ij}$. Then for $n > N$, $\|A_n - A\| < \epsilon$. Thus $\{A_n\} \rightarrow A$, and H is complete. Therefore H is a Hilbert space. □

Problem 3.4 (Luenberger)

Show that if $g(x_1, x_2, \ldots, x_n) = 0$, the normal equations possess a solution but it is not unique.

Solution to 3.4

First, we note that we can write the normal equations as:

$$\begin{bmatrix} \alpha_1 & \ldots & \alpha_n \end{bmatrix} \cdot \begin{bmatrix} (y_1 \mid y_1) & (y_1 \mid y_2) & \ldots & (y_1 \mid y_n) \\ (y_2 \mid y_1) & \cdot & \ldots & \cdot \\ \cdot & \cdot & \ldots & \cdot \\ \cdot & \cdot & \ldots & \cdot \\ (y_n \mid y_1) & \cdot & \ldots & (y_n \mid y_n) \end{bmatrix} = \begin{bmatrix} (x \mid y_1) \\ (x \mid y_2) \\ \cdot \\ \cdot \\ (x \mid y_n) \end{bmatrix}$$

$g(x_1, x_2, \ldots, x_n)$ is the determinant of the matrix G, above. From linear algebra we know that $g = 0$ means the columns of the matrix G are linearly dependent. This tells us that we have more unknowns than (independent) equations, or equivalently that either many vectors (all elements of a dimension $k \geq 1$ subspace) satisfy the normal equations , or (if the equations are inconsistent) that no vectors satisfy the normal equations. The Projection Theorem guarantees that a solution exists (i.e. the equations must be consistent), and thus an entire subspace of solutions exist. □

Problem 3.5 (Luenberger)

Find the linear function $x(t) = a + bt$ minimizing $\int_{-1}^{1} [t^2 - x(t)]^2 \, \mathrm{d}t$.

Solution to 3.5

If we consider ourselves working in $L_2[-1, 1]$, and considering the subspace $P = \{a + bt \mid a, b \in \mathbb{R}\}$, we see that this is a minimum norm problem from $t^2 \in L_2[-1, 1]$ to the subspace P. The projection theorem tells us that there is a unique solution, and we may find it using orthogonality. Our inner product is $\langle x|y\rangle = \int_{-1}^{1} x(t)y(t) \, \mathrm{d}t$. We could alternatively use calculus, but this does not give us the existence statement.

Method 1 Let $p = a + bt$ be the optimal affine function. The projection theorem tells us that $t^2 - p \perp P$, i.e. $\langle t^2 - p|\alpha + \beta t\rangle = 0$ for all $\alpha, \beta \in \mathbb{R}$.

$$\begin{aligned} 0 &= \langle t^2 - p|\alpha + \beta t\rangle = \int_{-1}^{1} (t^2 - a - bt)(\alpha + \beta t) \, \mathrm{d}t \\ &= \alpha \left[\frac{1}{3}t^3 - \frac{b}{2}t^2 - at\right]_{-1}^{1} + \beta \left[\frac{1}{4}t^4 - \frac{b}{3}t^3 - \frac{a}{2}t^2\right]_{-1}^{1} \\ &= \alpha \left[\frac{2}{3} - 2a\right] - \frac{2}{3}\beta b. \end{aligned}$$

Therefore $a = \frac{1}{3}$ and $b = 0$.

Method 2 Using simple calculus, we have $f(a, b) = \int_{-1}^{1} [t^2 - a - bt]^2 \, \mathrm{d}t$, so an extremum point must obey the twin equations $\mathrm{d}f/\mathrm{d}a = 0$, $\mathrm{d}f/\mathrm{d}b = 0$. We have

$$f(a, b) = \int_{-1}^{1} t^4 - 2bt^3 + (b^2 - 2a)t^2 + 2abt + a^2 \, \mathrm{d}t = \frac{2}{5} + \frac{2}{3}\left(b^2 - a\right) + 2a^2.$$

$$\begin{aligned} 0 = \frac{\mathrm{d}f}{\mathrm{d}a} &= -\frac{4}{3} + 4a \implies a = \frac{1}{3}. \\ 0 = \frac{\mathrm{d}f}{\mathrm{d}b} &= \frac{4}{3}b \implies b = 0. \end{aligned}$$

This is the only extremal point, and it is easy to show that it's a minimum: $\frac{\mathrm{d}^2 f}{\mathrm{d}a^2} = 4$, $\frac{\mathrm{d}^2 f}{\mathrm{d}b^2} = 0$.

So the (unique) solution is $x(t) = \frac{1}{3}$. □

Problem 3.6 (Luenberger)

Given $x \in L^2[0, 1]$, we seek a polynomial p of degree less than or equal to n which minimizes $\int_0^1 |x(t) - p(t)|^2 \, \mathrm{d}t$ while obeying the restriction that $\int_0^1 p(t) \, \mathrm{d}t = 0$.

(a) Show that the problem has a unique solution.

(b) Show that we may solve the problem via first finding q of degree $\leq n$ which minimizes $\|x - q\|^2$ and then finding p of degree $\leq n$ which minimzes $\|q - p\|^2$ while maintaining the constraint $\int_0^1 p(t) \, \mathrm{d}t = 0$.

Solution to 3.6

(a) Let Q be the space of polynomials of degree n or less. Let $P \subset Q$ be the set of polynomials which integrate to zero, i.e. $p_0 \in P \implies \int_0^1 p_0(t)\,\mathrm{d}t$. N is a closed subspace of $L_2[0,1]$, and P is a closed subspace of Q (and hence of $L_2[0,1]$ also). Obviously $L_2[0,1]$ is a Hilbert space, so the projection theorem tells us that there is a unique $p \in P$ such that $\forall p_0 \in P$, $\|x-p\| \le \|x-p_0\|$. Note that $\int_0^1 |x(t)-q(t)|^2\,\mathrm{d}t = \|x-q\|^2$, and minimizing the norm is equivalent to minimizing its square.

(b) Choose q and p as stated in the problem. By the projection theorem, we have that $x-q \perp Q$ and $q-p \perp P$. We want to show that $x-p \perp P$, so that the projection theorem tells us that $p \in P$ minimizes $\|x-p\|^2$. Since $0 \in P, Q$, we have the twin statements that

$$\forall q_0 \in Q, \quad \langle x-q|q_0\rangle \qquad \text{and} \qquad \forall p_0 \in P, \quad \langle q-p|p_0\rangle = 0.$$

Since for any $p_0 \in P$, we have $p-p_0 \in P \subset Q$, we have $\langle x-q|p-p_0\rangle = 0$ and $\langle q-p|p-p_0\rangle = 0$. Taking the difference of these two inner products gives our result, $\langle x-p|p-p_0\rangle = 0$, or $x-p \perp P$.

□

Problem 3.9 (Luenberger)

Prove $S^{\perp\perp} = \overline{[S]}$.

Solution to 3.9

As usual, we prove equality of the two sets by first showing that $S^{\perp\perp} \supseteq \overline{[S]}$, and then that $S^{\perp\perp} \subseteq \overline{[S]}$.

- $S^{\perp\perp} \supseteq \overline{[S]}$

  By Prop. 1, part 1 (Luenberger, page 52), we know $S^{\perp\perp}$ is a closed subspace. Part 2 of the same proposition tells us that $S \subset S^{\perp\perp}$. By definition, $\overline{[S]}$ is the smallest closed subspace containing S. This means every closed subspace containing S also contains $\overline{[S]}$. Thus, $S^{\perp\perp} \supseteq \overline{[S]}$ as desired.

- $S^{\perp\perp} \subseteq \overline{[S]}$

  It suffices to show that every closed subspace containing S must also contain $S^{\perp\perp}$. Let $B \supset S$ be a closed subspace. Then part 3 of Prop. 1 tells us $B^{\perp} \subset S^{\perp}$. Applying part 3 again yields $S^{\perp\perp} \subset B^{\perp\perp}$. But B is a closed subspace of a Hilbert space, and thus according to Theorem 1 (page 53) $B = B^{\perp\perp}$. So $S^{\perp\perp} \subset B$ for every closed B containing S. Thus $S^{\perp\perp}$ lies in the intersections of such B and is contained in $\overline{[S]}$, as desired.

□

Problem 3.24 (Luenberger)

Let K be a closed, convex set in a Hilbert space H, and let $x \in H$, $x \notin K$ be given. Then there is a unique $k_0 \in K$ such that $\|x-k_0\| \le \|x-k\|$ for all $k \in K$. Show that this is not true in an arbitrary Banach space.

Solution to 3.24

We will prove this via a counter-example. We choose $X = \ell_\infty$, the Banach space of bounded sequences. The sup-norm $\|\cdot\|_\infty$ is not derived from an inner product. We can show this from the parallelogram law: define $x = (1,0,0,\ldots)$ and $y = (0,1,0,\ldots)$. Then $\|x+y\|_\infty^2 + \|x-y\|_\infty^2 = 1+1 = 2$, which differs from $2\|x\|_\infty^2 + 2\|y\|_\infty^2 = 4$.

Let $K = [x] = (\alpha, 0, \ldots)$, $\alpha \in \mathbb{R}$, that is, the set of elements with zeros everywhere after the first element. Let $y = (0,1,0,\ldots)$. Then $\|y-k\|_\infty \ge 1$ for all $k \in K$, but $\|y-k\|_\infty = 1$ for all $k = (\alpha, 0, \ldots)$, $\alpha \in [-1,1]$. So the minimizer to K is not unique. □