Solutions to Homework #3, Math 116

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Problem 1 (Goroff)

In each of the following cases, determine the supremum of f over its domain D. If there are points $x \in D$ for which $\sup f = f(x)$, find them and hence determine $\max f$.

Solution to 1

1. (a) $f(x) = 2x - 1$, $D = (0, 1)$. Here f is an affine function with positive slope, so it does not achieve a max on the open set $(0, 1)$, and its supremum is 1.

   (b) $f(x) = 3 - x^2$, $D = \mathbb{R}$. Actually D was unspecified in this problem, so we assume it to be the whole applicable domain of definition. Obviously $\sup f = 3$, with $\max f = f(0) = 3$.

   (c) $f(x) = \sin x$, $D = \mathbb{R}$. $\sup f = 1$, $\max f = f\left(\frac{\pi}{2} + 2\pi n\right)$.

   (d) $f(x) = x \sin(x)$, $D = \mathbb{R}$. f grows unbounded as $|x| \to \infty$, so $\sup f$ DNE (does not exist). It is incorrect to write $\sup f = \infty$, because ∞ is not a real number. However, I did not deduct points for this oversight.

   (e) $f(x) = 1/x$, $D = \mathbb{R}$. Again, $\sup f$ DNE (does not exist), since $\lim\limits_{x \to 0^+} f(x) \to \infty$.

   (f) $f(x) = \left\{\begin{matrix} x & : & 0 < x < 1 \\ 1 - x & : & 1 < x < 2 \end{matrix}\right\}$, $D = (0, 1) \cup (1, 2)$. Here $\sup f = 1$, and this is not achieved on the domain D.

   (g) $f(x) = 3x^2 - 2x - 6$, $D = [0, 4]$. Here we have a compact domain D and a continuous function f. So we will achieve a maximum somewhere, either in the interior or at an endpoint. Simple calculus show that $\frac{\mathrm{d}}{\mathrm{d}x} f = 6x - 2 = 0 \implies x = \frac{1}{3}$ is an extremum point. But that's the minimum point, $f\left(\frac{1}{3}\right) = -\frac{19}{3}$, while $f(0) = -6$ and $f(4) = 34 = \max f$.

$\square$

Problem 2 (Goroff)

Consider the function space X which consists of all functions of a single variable of the form $f(x) = a \cdot x + b$ for all real x. Let J denote the functional on X defined by

$$J[f] = \int_0^1 f \mathrm{d}x$$

Solution to 2

1. (a) Is every f in X a linear function on the real line according to our definition?

No. For a function to be linear according to our definition, it must be the case that $f(\alpha x+\beta y) = \alpha f(x) + \beta f(y)$. Let $f(x) = ax + b$ for some $b \neq 0$ Then $f(\beta x) = a\beta x + b \neq \beta(ax + b)$, as $\beta b \neq b$, unless $\beta = 1$ or $b = 0$. One useful fact (which some of you used to answer this question) is that if f is linear, $f(0) = 0$ (Assume $f(0) = b$ for $b \neq 0$. But since f is linear, $f(0) = f(0 + 0) = f(0) + f(0) = 2b \neq b$. Contradiction, thus f linear implies $f(0) = 0$). Of course, not all functions that map zero to zero are linear (i.e. $f(0) = 0$ is a necessary condition for linearity, but not a sufficient condition.)

(b) Is X a vector space or what? What is its dimension? Can you find a basis? Another basis?

Yes, X is a vector space. In particular, it is a subset of the continuous functions or of the polynomials, which we know are vector spaces. So we just need to show that X is closed under scalar multiplication and vector addition. Let $f(x) = ax + b$, where $a, b \in R$. Then $\alpha f(x) = \alpha(ax + b) = \alpha ax + \alpha b$. But $\alpha a, \alpha b \in R$. Thus $\alpha f(x) \in X$. Let $g(x) = cx + d$. Then

$$\begin{aligned}(f + g)(x) &= f(x) + g(x) \\ &= (ax + b) + (cx + d) \\ &= (a + c)x + (b + d)\end{aligned}$$

Since $(a + c), (b + d) \in R$, $(f + g)(x) \in X$. Thus X is a subspace of the polynomials, and thus a vector space.

The easiest way to determine the dimension of X is to find a basis. I claim $\{1, x\}$ are a basis for X. First, we note than every vector $f(x)$ can be written as $b \cdot 1 + a \cdot x$. So $\{1, x\}$ span X. Also, it is clear that $ax + b = 0$ only when $a = 0, b = 0$. Thus $\{1, x\}$ are linearly independent. So by definition, $\{1, x\}$ is a basis. Since this basis has 2 elements, we know that the dimension of X is 2. A little work shows that another basis is $\{x + 1, 7\}$.

(c) Is X isometrically isomorphic to some familiar space? What does X need before you can answer?

To answer this question fully, we need to equip X with a norm. We'll define $||f|| = \sqrt{a^2 + b^2}$. To show this is a norm, we need to show:

- $||f|| = 0 \iff f = 0$
  We observe that $\sqrt{a^2 + b^2} = 0 \iff a = 0, b = 0$.
- $||\alpha f|| = |\alpha| \cdot ||f||$
  $\alpha f = \alpha ax + \alpha b$. Thus $||\alpha f|| = \sqrt{\alpha^2 a^2 + \alpha^2 b^2} = |\alpha|\sqrt{a^2 + b^2}$ as required.
- $||f + g|| \leq ||f|| + ||g||$
  Expanding each side we get:

  $$\sqrt{(a + c)^2 + (b + d)^2} \leq \sqrt{a^2 + b^2} + \sqrt{c^2 + d^2}$$

  This statement is identical to the triangle inequality in R^2 with the Euclidean norm. We know it holds for the Euclidean norm, thus it must hold here.

Thus $||f|| = \sqrt{a^2 + b^2}$ is a norm.

To show X is isometrically isomorphic to R^2, we'll explicitly define an isometric isomorphism $T : X \mapsto R^2$ as $T[f] = (a, b)$ for $f = ax + b$. First, I'll show this map is a linear bijection (i.e. isomorphism of vector spaces). Clearly T is onto, since any point (c, d) is mapped to by $f(x) = cx + d$. T is one-to-one since each f can be written in exactly one way as $ax + b$ and thus gets mapped to exactly one order pair (a, b). To see that T is linear, consider:

$$\begin{aligned} T(\alpha f + \beta g) &= (\alpha a + \beta c, \alpha b + \beta d) \\ &= (\alpha a, \alpha b) + (\beta c, \beta d) \\ &= \alpha(a, b) + \beta(c, d) \\ &= \alpha T(f) + \beta T(g) \end{aligned}$$

Thus T is an isomorphism of vector spaces.

Now we need to show $||T(f)|| = ||f||$ for all $f \in X$. For $f(x) = ax + b$ (as usual), $||T(f)|| = ||(a, b)|| = \sqrt{a^2 + b^2} = ||f||$. So T is an isometric isomorphism between X and R^2.

(d) Evaluate $J[f]$. Is J linear?

$$\begin{aligned} J[f] &= \int_0^1 f \mathrm{d}x \\ &= \int_0^1 ax + b \mathrm{d}x \\ &= \frac{ax^2}{2}|_0^1 + bx|_0^1 \\ &= \frac{a}{2} + b \end{aligned}$$

To see that J is linear, note $J[f + g] = \frac{a+c}{2} + (b+d) = J[f] + J[g]$ and $J[\alpha f] = \frac{\alpha a}{2} + \alpha b = \alpha J[f]$

(e) Can you find $sup\{J[f] | f \in X\}$?

No. We know for $f(x) = ax + b$, $J[f] = \frac{a}{2} + b$ which increase to ∞ as either a or b go to ∞. Thus there is no upper bound for J

(f) What about $inf\{J[f] | f \in X\}$?

Similarly, as $a, b \to -\infty$, $J[f] \to -\infty$. Thus there is no lower bound.

□

Problem 3 (Goroff)

Consider the set Y which consists of all functions of a single variable of the form $f(x) = a \cdot x$ for $0 \leq a \leq 1$. Let J denote the functional on Y defined by:

$$J[f] = \int_0^a ax \mathrm{d}x - a$$

for $f(x) = ax$.

Solution to 3

1. (a) Is Y a subspace of the space X of the previous problem?

No. It is easy to see that Y is not closed under vector addition. Let $f(x) = g(x) = x$. Then $(f + g)(x) = f(x) + g(x) = x + x = 2x$. But $2 > 1$, thus $(f + g)$ is not in Y. So Y is not a vector space – in particular, it is not a subspace of X in Problem 2.

Is J linear?

No. We evaluate

$$\begin{aligned} J[f] &= \int_0^a ax\mathrm{d}x - a \\ &= \frac{ax^2}{2}|_0^a - a \\ &= \frac{a^3}{2} - a \end{aligned}$$

To see that J isn't linear, let $f(x) = ax$, $g(x) = b(x)$. $J[f+g] = \frac{(a+b)^3}{2} - (a+b)$. But $J[f] + J[g] = \frac{a^3+b^3}{2} - (a+b)$. Thus J isn't linear.

(b) Before tyring to calculate $sup\{J[f] \,|f \in Y\}$, how do you know it must be finite and attained? Justify your answer by carefully applying an appropriate theorem.

From our evaluation of $J[f]$ above, we can view J as a function of a. $a \in [0,1]$, and $\frac{a^3}{2} - a$ is just a polynomial in a and thus is continuous. a can only take values from the closed interval $[0,1]$. So $J(a)$ is a continuous function on a compact set (note $[0,1]$ is a closed, bounded subset of a finite dimensional vector space and thus is compact). Thus the Weierstrauss theorem tells us that $\exists a \in [0,1]\, s.t J(a)$ is maximized. Since $J[f] = J(a)$, we know $J[f]$ attains it's maximum on Y.

(c) Can you find $sup\{J[f] \,|f \in Y\}$? As we noted above, we can treat J as a function of a. As is standard practice in single variable calculus, we differentiate J w.r.t a and set the derivative equal to zero.

$$\begin{aligned} J'(a) &= \frac{3a^2}{2} - 1 \\ 0 &= \frac{3a^2}{2} - 1 \\ \frac{2}{3} &= a^2 \\ a &= +\sqrt{\frac{2}{3}} \end{aligned}$$

So we need to check this critical point and the two endpoints. $J(0) = 0, J(1) = -\frac{1}{2}, J(+\sqrt{\frac{2}{3}}) = -\frac{2\sqrt{6}}{9}$. Thus the max value of J is 0. This maximum occurs at $f(x) = 0$.

(d) What about $inf\{J[f] \,|f \in Y\}$?

Again, we look at the values of J on the boundary and at critical point. J attains a minimum of $-\frac{2\sqrt{6}}{9}$ at $f(x) = +\sqrt{\frac{2}{3}}x$.

□

Problem 4 (Goroff)

Consider the sequence of functions $f_n(x) = nx(1-x)^n$ on the domain $D = [0,1]$.

Solution to 4

1. (a) Show that the sequence $\{f_n\}$ convereges pointwise to some f. What is it?

The sequence converges pointwise to the zero function. Consider nx and $(1-x)^n$: Their product equals $f_n(x)$, but as $n \to \infty$, $nx \to \infty$ and $(1-x)^n$ $to0$. Do they grow at the same of different rates? $(1-x)^n$ is a power law, so it converges to 0 exponential with n while nx only grows linearly with n.

(b) How do you know that f_n achieves a maximum on D? Find the maxima.

Well, D is compact and f_n is continuous, so Weierstrass tells us that a max and min exist on D. Now we use calculus:

$$0 = \frac{\mathrm{d}}{\mathrm{d}x} f_n = n(1-x)^n - n^2 x(1-x)^{n-1}$$

$$\therefore 0 = (1-x) - nx \implies x = \frac{1}{n+1} \text{ is an extremum.}$$

The endpoints $f_n(0) = 0 = f_n(1)$ are minima, and $f_n\left(\frac{1}{n+1}\right) = \frac{n}{n+1}\left(1 - \frac{1}{n+1}\right)^n$.

(c) To what value would you expect the sequence $\left\{f_n\left(\frac{1}{n+1}\right)\right\}$ to converge? Check that the actual limit is e^{-1}.

We might expect the limit to be 0, since the sequence of functions $\{f_n\}$ converges to the zero function. However, remember that $\lim_{n\to\infty}\left(1+\frac{r}{n}\right)^n = \mathrm{e}^r$. We see that

$$\lim_{n\to\infty} f_n\left(\frac{1}{n+1}\right) = \lim_{n\to\infty} \frac{n}{n+1}\left(1 - \frac{1}{n+1}\right)^n = \lim_{n\to\infty}\left(1 - \frac{1}{n+1}\right)^{n+1} = \frac{1}{\mathrm{e}}.$$

(d) Does the sequence $\{f_n\}$ converge in $C[0,1]$? That is, does it converge uniformly?

If $\{f_n\}$ converged uniformly, then it would have to converge to a continuous function. No problem there. However, the uniform norm is $\|f - f_n\|_\infty = \sup\limits_{x\in[0,1]} |f(x) - f_n(x)|$. Since $f(x) = 0$ and $f_n(x) \geq 0$, we see that $\|f - f_n\|_\infty = \sup\limits_{x\in[0,1]} f_n(x)$. However, we have $\max f_n \to 1/\mathrm{e}$, not 0, so $\lim\limits_{n\to\infty} \|f - f_n\|_\infty = 1/\mathrm{e}$, and we do not have uniform convergence.

$\square$