

Solutions to Homework #2, Math 116

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Problem 2.12 (Luenberger)

Suppose $x = \{\xi_1, \xi_2, \dots, \xi_n, \dots\} \in l_{p_0}$ for some $p_0, 1 \leq p_0 \leq \infty$.

- **Part 1** Show that $x \in l_p$ for all $p \geq p_0$.
- **Part 2** Show that $\|x\|_\infty = \lim_{p \rightarrow \infty} \|x\|_p$.

Solution to 2.12

• Part 1

Explicitly, we want to show $\sum_{i=1}^{\infty} |\xi_i|^{p_0} < \infty$ implies $\sum_{i=1}^{\infty} |\xi_i|^p < \infty$ for all $p \geq p_0$. However, it is important to realize that it is often not the case that $\sum_{i=1}^{\infty} |\xi_i|^p < \sum_{i=1}^{\infty} |\xi_i|^{p_0}$. We will prove that in the special case where $|\xi_i| < 1 \forall i$, it is in fact true that $\sum_{i=1}^{\infty} |\xi_i|^p < \sum_{i=1}^{\infty} |\xi_i|^{p_0}$ whenever $p \geq p_0$. Then we will prove that any element of l_{p_0} contains only finitely many terms with absolute value greater than or equal to 1. As we know that a finite number of terms does not effect the convergence of a sum, this suffices to prove the claim.

Claim 1: $\sum_{i=1}^{\infty} |\xi_i|^p \leq \sum_{i=1}^{\infty} |\xi_i|^{p_0}$ whenever $p \geq p_0$ and $|\xi_i| < 1 \forall i$.

First we note that for any positive real number $x < 1$ and $k > 1$, $x^k < x$:

$$\begin{aligned}x &< 1 \\x^{k-1} &< 1^{k-1} \\x^{k-1} &< 1 \\x \cdot x^{k-1} &< x \cdot 1 \\x^k &< x\end{aligned}$$

as desired. (Note: as this is a simple fact from elementary algebra, it could be stated without proof.) Thus, since $|\xi_i| < 1 \forall i$, $|\xi_i|^{p-p_0} < |\xi_i| < 1$ for $p > p_0$, and $|\xi_i|^p < |\xi_i|^{p_0}$ (by multiplication by $|\xi_i|^{p_0}$). So $\sum_{i=1}^{\infty} |\xi_i|^p < \sum_{i=1}^{\infty} |\xi_i|^{p_0}$ as each term of the first sum is smaller than the corresponding term in the second sum. Equality in the statement holds when $p = p_0$.

Claim 2: Any element of l_{p_0} contains only finitely many terms with absolute value greater than or equal to 1.

Assume this is not the case. Let $x = \{\xi_1, \xi_2, \dots, \xi_n, \dots\} \in l_{p_0}$ be an element with infinitely many terms ≥ 1 . Let $I := \{i \mid |\xi_i| \geq 1\}$. $\sum_{i \in I} |\xi_i|^p \geq \sum_{i \in I} |\xi_i|$ as each term on the left is greater than or equal to the corresponding term on the right. The sum on the right has infinitely many elements

≥ 1 . Thus for any $\epsilon < 1$, there is no N such that for $n, m > N$, $|S_n - S_m| < \epsilon$. Thus the sequence of partial sums is not Cauchy. Since every convergence sequence is Cauchy, this tells us that the partial sums do not converge, and $\sum_{i \in I} |\xi_i| \geq \infty$. Since $\sum_{i \in I} |\xi_i|^{p_0} \geq \sum_{i \in I} |\xi_i|$ and $\sum_{i=1}^{\infty} |\xi_i|^{p_0} > \sum_{i \in I} |\xi_i|^{p_0}$, this shows x is not in l_{p_0} .

To finish the proof, we write $\sum_{i=1}^{\infty} |\xi_i|^p = \sum_{i \in I} |\xi_i|^p + \sum_{i \in I^C} |\xi_i|^p$, where I is defined as in Claim 2, and I^C refers to the complement of I in $Z_{>1}$. We know $|I| < \infty$ and $\sum_{i \in I} |\xi_i|^p = B < \infty$. Thus:

$$\begin{aligned} \sum_{i=1}^{\infty} |\xi_i|^p &= B + \sum_{i \in I^C} |\xi_i|^p \\ &< B + \sum_{i \in I^C} |\xi_i|^{p_0} \\ &< B + \sum_{i=1}^{\infty} |\xi_i|^{p_0} \\ &< \infty \end{aligned}$$

because x is in l_{p_0} by assumption. Therefore $x \in l_p$ for $p \geq p_0$.

- **Part 2** First we need to show that $x \in l_{p_0}$ implies $x \in l_{\infty}$. Then we need to show that in fact $\|x\|_{\infty} = \lim_{p \rightarrow \infty} \|x\|_p$.

$x \in l_{\infty}$ means that $\exists N < \infty$ s.t. $\xi_i < N \forall i$. Let $M = \sup_i |\xi_i|$. We note that for $M > 1$, $M < \sum_{i=1}^{\infty} |\xi_i|^{p_0} < \infty$. Thus M is finite. To show that $M = |\xi_i|$ for some i , consider the contrary. If $M \neq |\xi_i|$ for all i , then $\exists \{x_{i_k}\}$ s.t. $\{x_{i_k}\} \rightarrow M$ by the definition of sup. Thus there are infinitely many ξ_i within ϵ of M . If $M > 0$, this implies that $\sum_{i=1}^{\infty} |\xi_i|^{p_0} > \infty$ since infinitely many of its terms are within ϵ of M^{p_0} . Thus M must actually be attained. Hence $x \in l_{\infty}$ and $\|x\|_{\infty} = M$.

To show that $\|x\|_{\infty} = \lim_{p \rightarrow \infty} \|x\|_p$, it suffices to show that for $x \neq \theta$ (i.e. the sequence of all zeros) $\lim_{p \rightarrow \infty} \frac{\|x\|_p}{M} = 1$. To prove this, first we note that there are only finitely many i such that $|\xi_i| = M$. (As usual, if not x is not in l_{p_0} .) Suppose there are k such terms, and let $I = \{i.s.t. |\xi_i| < M\}$

$$\begin{aligned} L &= \lim_{p \rightarrow \infty} \|x\|_p \\ &= \lim_{p \rightarrow \infty} \left(\sum_{i=1}^{\infty} |\xi_i|^p \right)^{\frac{1}{p}} \\ L^p &= \lim_{p \rightarrow \infty} \left(\sum_{i=1}^{\infty} |\xi_i|^p \right) \\ \left(\frac{L}{M} \right)^p &= \lim_{p \rightarrow \infty} \left(\sum_{i=1}^{\infty} \left(\frac{|\xi_i|}{M} \right)^p \right) \\ &= \lim_{p \rightarrow \infty} \left(\sum_{i \in I} \left(\frac{|\xi_i|}{M} \right)^p \right) + \lim_{p \rightarrow \infty} \left(\sum_{j=1}^k \left(\frac{M}{M} \right)^p \right) \\ &= 0 + k \end{aligned}$$

This implies $\lim_{p \rightarrow \infty} \frac{\|x\|_p}{M} = \lim_{p \rightarrow \infty} k^{\frac{1}{p}} = 1$. Hence $\|x\|_{\infty} = \lim_{p \rightarrow \infty} \|x\|_p$ as desired.

□

Problem 2.13 (Luenberger)

Prove that the normed space $D[a, b]$ is complete.

Solution to 2.13

To show that a space is complete, we need to show that every Cauchy sequence converges w.r.t the given norm. For $D[a, b]$ we'll use the norm $\|f\| = \max_{t \in [a, b]} |f(t)| + \max_{t \in [a, b]} |f'(t)|$. Also, $\|f\|_C$ means f 's norm in $C[a, b]$. Let $\{f_n\}$ be a Cauchy sequence in D .

- **Claim 1:** $\{f_n\}$ is Cauchy w.r.t $\|\cdot\|_C$ and thus converges to some f in $C[a, b]$ (as C is complete.)

Fix $\epsilon > 0$. Then $\exists N$ s.t $\|f_n - f_m\| < \epsilon$ for all $n, m > N$. But $\|f_n - f_m\|_C < \|f_n - f_m\|$, as $\max_{t \in [a, b]} |f'_n(t) - f'_m(t)| > 0$. Thus $\{f_n\}$ is also Cauchy w.r.t $\|\cdot\|_C$.

- **Claim 2:** $\{f'_n\}$ is Cauchy w.r.t $\|\cdot\|_C$ and thus converges to some f' in $C[a, b]$.

We just note that $\max_{t \in [a, b]} |f_n(t) - f_m(t)| > 0$ and apply the same argument as above.

- **Claim 3:** f' is in fact the derivative of f .

We know (from the fundamental theorem of calculus) that $f_n(t) = \int_a^t f'_n(x) dx$. Since $\{f_n\} \rightarrow f$ in C , $\{f_n\} \rightarrow f$ uniformly as well.

$$\begin{aligned} f(t) &= \lim_{n \rightarrow \infty} f_n(t) \\ &= \lim_{n \rightarrow \infty} \int_a^t f'_n(x) dx \\ &= \int_a^t \lim_{n \rightarrow \infty} f'_n(x) dx \quad \text{as } \{f'_n\} \rightarrow f' \text{ uniformly.} \\ &= \int_a^t f'(x) dx \\ f(t) &= \int_a^t f'(x) dx \\ \Rightarrow \frac{d}{dt} [f(t)] &= f'(t) \quad \text{by the fund. thm. of calculus} \end{aligned}$$

This completes the proof of Claim 3. It is important to note that you can only interchange the limit and integral because $\{f'_n\} \rightarrow f'$ uniformly. (Most other methods people used to prove this 3rd claim were not completely correct, although often no points were deducted if you correctly stated the claim and attempted to prove it in a reasonable fashion.)

Now all that remains is to show that in fact, $\{f_n\} \rightarrow f$ in D . To see this, note that $\|f - f_n\| = \|f - f_n\|_C + \|f' - f'_n\|_C$. Fix $\epsilon > 0$. From claims 1 and 2, we know we can choose L s.t $\|f - f_l\|_C < \frac{\epsilon}{2}, \forall l > L$ and M s.t $\|f' - f'_m\|_C < \frac{\epsilon}{2}, \forall m > M$. Set $N = \max\{L, M\}$. Then for $n > N$, $\|f - f_n\| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$. Thus $\{f_n\} \rightarrow f$ in D and D is complete. \square

Problem 2.14 (Luenberger)

Two vector spaces, X and Y , are said to be isomorphic if there is a one-to-one mapping T of X onto Y such that $T(\alpha_1 x_1 + \alpha_2 x_2) = \alpha_1 T(x_1) + \alpha_2 T(x_2)$. Show that any real n -dimensional space is isomorphic to E^n .

Solution to 2.14

The easiest way to do this is to explicitly define such an isomorphism. Let $\{e_1, \dots, e_n\}$ be the standard

basis on E^n . Let $\{x_1, \dots, x_n\}$ be a basis on the n -dimensional space X . Note, we know X has a basis because every finite dimensional space has a basis. Let $T : X \rightarrow E^n$ be a map such that for $x = \sum_{i=1}^n \alpha_i x_i$, $T(x) = \sum_{i=1}^n \alpha_i e_i$. We need to show this map is well-defined, one-to-one, onto, and linear.

- T is well-defined because any x can be written as a linear combination of basis elements in exactly one way.
- To see that T is one-to-one, assume $x, y \in X$ s.t. $T(x) = T(y)$. Since e_i is a basis of E^n there is only one way to write $T(x) = T(y) = \sum_{i=1}^n \alpha_i e_i$. In order for this to be the case it must be that $x = \sum_{i=1}^n \alpha_i x_i$ and $y = \sum_{i=1}^n \alpha_i x_i$. Thus $x = y$ and T is one-to-one.
- To see that T is onto, we simply note that any linear combination $\sum_{i=1}^n \alpha_i e_i$ in E^n is hit by $\sum_{i=1}^n \alpha_i x_i$ in X .
- Now we just need to show that $T(\alpha x + \beta y) = \alpha T(x) + \beta T(y)$. Let $x = \sum_{i=1}^n \alpha_i x_i, y = \sum_{i=1}^n \beta_i x_i$. Then

$$\begin{aligned}
 \alpha x + \beta y &= \alpha \sum_{i=1}^n \alpha_i x_i + \beta \sum_{i=1}^n \beta_i x_i \\
 &= \sum_{i=1}^n (\alpha \alpha_i + \beta \beta_i) x_i \\
 T(\alpha x + \beta y) &= \sum_{i=1}^n (\alpha \alpha_i + \beta \beta_i) e_i \\
 &= \alpha \sum_{i=1}^n \alpha_i e_i + \beta \sum_{i=1}^n \beta_i e_i \\
 &= \alpha T(x) + \beta T(y)
 \end{aligned}$$

□

Problem 2.15 (Luenberger)

Show that any normed linear space X is isometrically isomorphic to a dense subset of a Banach space \widehat{X} .

Solution to 2.15

- (a) Let \overline{X} be the set of Cauchy sequences $\{x_n\}$ from X with addition and scalar multiplication defined coordinate-wise. We want to show that \overline{X} is a linear space. Let $\alpha, \beta \in \mathbb{R}$, $\{x_i\}, \{y_i\} \in \overline{X}$. We want to show that $\alpha\{x_i\} + \beta\{y_i\}$ is a Cauchy sequence, i.e. is in \overline{X} . We have $\alpha\{x_i\} + \beta\{y_i\} = \{\alpha x_i + \beta y_i\}$. Since $\{x_i\}$ and $\{y_i\}$ are Cauchy, given $\epsilon > 0$ we choose $N_x, N_y \in \mathbb{N}$ s.t. $\forall n, m > N_x$ and $\forall p, q > N_y$ we have $|x_n - x_m| < \epsilon/2|\alpha|$ and $|y_p - y_q| < \epsilon/2|\beta|$. Then by choosing $N = \max\{N_x, N_y\}$ we have for all $n, m > N$

$$|\alpha x_n + \beta y_n - \alpha x_m - \beta y_m| \leq |\alpha| \cdot |x_n - x_m| + |\beta| \cdot |y_n - y_m| < \epsilon/2 + \epsilon/2 = \epsilon.$$

So $\alpha\{x_i\} + \beta\{y_i\} = \{\alpha x_i + \beta y_i\}$ is Cauchy and \overline{X} is a linear space.

(b) For $y = \{x_i\}$, define $\|y\| = \sup_n \|x_n\|$. We want to show that $(\overline{X}, \|\cdot\|)$ is a normed space. We proceed via the definition of a norm. To avoid notational confusion, I'll write $\|y\|_{\overline{X}}$ for the norm of the Cauchy sequence elements of \overline{X} and I'll use $\|x_n\|_X$ for the norm in X . So we have for $y = \{x_n\} \in \overline{X}$, $\|y\|_{\overline{X}} = \|\{x_n\}\|_{\overline{X}} = \sup_n \|x_n\|_X$.

1. Since $\|\cdot\|_X$ is a norm, we have $\|\cdot\|_X \geq 0$, and $\|x_n\|_X = 0 \implies x_n = 0$. Consequently, we have $\|y\|_{\overline{X}} = \sup_n \|x_n\|_X \geq 0$, and $\|y\|_{\overline{X}} = 0 \implies \|x_n\|_X = 0 \implies x_n = 0$ for all n . Hence $y = \{x_i\} = \{0, 0, \dots\}$.
2. Let $\alpha \in \mathbb{R}$ be given. Then we have

$$\|\alpha y\|_{\overline{X}} = \|\{\alpha x_i\}\|_{\overline{X}} = \sup_n \|\alpha x_n\|_X = \sup_n |\alpha| \cdot \|x_n\|_X = |\alpha| \cdot \|y\|_{\overline{X}}.$$

3. Let $z = \{\xi_i\} \in \overline{X}$ be given. We want the triangle inequality:

$$\begin{aligned} \|y + z\|_{\overline{X}} &= \|\{x_i + \xi_i\}\|_{\overline{X}} = \sup_n \|x_n + \xi_n\|_X \leq \sup_n (\|x_n\|_X + \|\xi_n\|_X) \\ &\leq \sup_n \|x_n\|_X + \sup_m \|\xi_m\|_X = \|y\|_{\overline{X}} + \|z\|_{\overline{X}}. \end{aligned}$$

(c) Let M be the subspace of Cauchy sequences which converge to zero. Define $\widehat{X} = \overline{X}/M$. We want to show that \widehat{X} is a Banach space with norm $\left\| [\{x_n\}] \right\|_{\widehat{X}} = \inf_{m \in M} \|\{x_n\} + m\|_{\overline{X}}$.

First, we show that M is closed wrt $\|\cdot\|_{\overline{X}}$. Let $\epsilon > 0$ and $y \in \overline{X}$ a closure point of M be given. So we may choose $z = \{\xi_i\} \in M$ such that $\|z - y\|_{\overline{X}} < \epsilon/2$. We want to show that $y \in M$, i.e. that $x_n \rightarrow 0$. To achieve this, we'll show that there exists an N such that for all $n > N$, $\|x_n\|_X < \epsilon$. Since $z = \{\xi_i\}$ is Cauchy, we may choose $N > 0$ such that for all $n > N$ we have $\|\xi_n\|_X < \epsilon/2$. Then what is $\|x_n\|_X$ for $n > N$? We have

$$\|x_n\|_X = \|x_n - \xi_n + \xi_n\|_X \leq \|x_n - \xi_n\|_X + \|\xi_n\|_X < \epsilon/2 + \epsilon/2 = \epsilon.$$

So $\|x_n\|_X$ gets arbitrarily small with increasing N . Hence $y \in M$ and M is closed.

Since M is a closed subspace of \overline{X} , proposition 1 from section 2.14 gives us that $\widehat{X} = \overline{X}/M$ is a Banach space.

Consider $T : X \rightarrow \widehat{X}$ defined via for $x \in X$, $T(x) = [x, x, x, \dots]$. T is an isomorphism since for $\{\alpha_i\} \subset \mathbb{R}$, $\{x_i\} \subset X$ we have

$$T\left(\sum_i \alpha_i x_i\right) = \left[\left\{ \sum_i \alpha_i x_{i1}, \sum_i \alpha_i x_{i2}, \dots \right\} \right] = \sum_i \alpha_i [x_i] = \sum_i \alpha_i T(x_i).$$

Clearly T is an isometry. We also see that $T(X)$ is the set of equivalence classes of all Cauchy sequences which converge.

□

Problem 2.17 (Luenberger)

Let M be an m -dimensional subspace of an n -dimensional vector space X . Show that X/M is $(n - m)$ -dimensional.

Solution to 2.17

Choose a basis of X $\{x_1, x_2, \dots, x_n\}$ such that $\{x_1, x_2, \dots, x_m\}$ is a basis for M . We will show that $\{[x_{m+1}], [x_{m+2}], \dots, [x_n]\}$ serves as a basis for X/M . Clearly these elements are independent since the set $\{x_i\}$ is independent in X . For all i , $[x_i] \in X/M$, so $\text{span}\{[x_{m+1}], [x_{m+2}], \dots, [x_n]\} \subset X/M$. We now want to show the other inclusion. Let $[y] \in X/M$ be given. Write out y in terms of its basis in X : $y = \sum_{i=1}^n \alpha_i x_i$, $\{\alpha_i\} \subset \mathbb{R}$. Now $\{x_i, i \leq m\} \subset M$, so $\sum_{i=1}^m \alpha_i x_i \in M$. We know that for $[x], [y] \in X/M$, $[x] = [y] \iff x - y \in M$. Hence $[y] = [\sum_{i=1}^n \alpha_i x_i] = [\sum_{i=m+1}^n \alpha_i x_i] = \sum_{i=m+1}^n \alpha_i [x_i] \in \text{span}\{[x_{m+1}], [x_{m+2}], \dots, [x_n]\}$, and thus $X/M = \text{span}\{[x_{m+1}], [x_{m+2}], \dots, [x_n]\}$. Since $\{[x_{m+1}], [x_{m+2}], \dots, [x_n]\}$ is thereby a basis for X/M , and it has $n - m$ elements. The dimension of a finite-dimensional vector space equals the cardinality of any of its basis sets. \square

Problem 2.19 (Luenberger)

Let X be a vector space with semi-norm $|\cdot|$, and consider $M = \{x : |x| = 0\}$. Show that the space X/M with $\|[x]\| = \inf_{m \in M} |x + m|$ is normed.

Solution to 2.19

Note that the semi-norm $|\cdot|$ fails as a norm only if $\exists x \neq 0$ such that $|x| = 0$. Let $M = \{x : |x| = 0\}$; consider the quotient space X/M with operation $\|[x]\| = \inf_{m \in M} |x + m|$ is normed. Note that $\|\cdot\|$ works on objects in X/M , i.e. equivalence classes of objects in X , while the semi-norm works on objects in X . There are three requirements to be a norm:

- (a) Let $[x] \in X/M$ be given. We want to show that $\|[x]\| \geq 0$ and that if $\|[x]\| = 0$ then $[x] = [0]$. $\|[x]\| = \inf_{m \in M} |x + m| \geq 0$. Observe that $\forall m \in M$, we have $|m| = 0$, so that

$$|x| = |x + m - m| \leq |x + m| + |-m| = |x + m| + |-1| \cdot |m| \leq |x + m| \leq |x| + |m| = |x|.$$

Consequently $\|[x]\| = \inf_{m \in M} |x + m| = |x|$.

$\|[x]\| = 0 \implies |x| = 0$, so $x \in M$. Hence $[x] = [0]$.

- (b) Let $\alpha \in \mathbb{R}$ be given.

$$\|[\alpha x]\| = \|\alpha[x]\| = \inf_{m \in M} |\alpha x + m| = \inf_{m' \in M} |\alpha x + \alpha m'| = \inf_{m' \in M} |\alpha| \cdot |x + m'| = |\alpha| \cdot \|[x]\|.$$

- (c) Note that for $m_1, m_2 \in M$, $0 \leq |m_1 + m_2| \leq |m_1| + |m_2| = 0$, so $m_1 + m_2 \in M$. So M is closed under addition, and since $0 \in M$, every element in M can be expressed as the sum of two elements in M . Let $x, y \in X$ be given. Then we have

$$\begin{aligned} \|[x] + [y]\| &= \|[x + y]\| = \inf_{m \in M} |x + y + m| = \inf_{m_1, m_2 \in M} |x + y + m_1 + m_2| \\ &\leq \inf_{m_1 \in M} |x + m_1| + \inf_{m_2 \in M} |y + m_2| = \|[x]\| + \|[y]\|. \end{aligned}$$

We conclude that $\|\cdot\|$ is a norm on X/M . \square