

Yes, you are quite correct that  $\bar{X}$  need not be complete. Notice, by the way, that the elements of your sequence are all equivalent to one another in  $\bar{X}/M = \hat{X}$ . To see directly that  $\hat{X}$  is complete, start by realizing that the norm in that space is  $\|\hat{x}\| = \lim_{n \rightarrow \infty} \|x_n\|$  where  $(x_n) \sim \hat{x}$ . Now suppose that  $(\hat{x}_n)$  is a Cauchy sequence in  $\hat{X}$ . Because  $X$  is isometrically isomorphic to a dense subset  $W$  of  $\hat{X}$ , you can find, for every  $n$ , a  $(\hat{z}_n)$  in  $W$  such that  $\|\hat{x}_n - \hat{z}_n\| < \frac{1}{n}$ . Using the triangle inequality, you can then show that  $(\hat{z}_n)$  is Cauchy, too. So take a Cauchy sequence  $(z_n)$  back in  $X$  which corresponds to this  $(\hat{z}_n)$  under the isomorphism. Then let  $\hat{x} \in \hat{X}$  be the class to which  $(z_n)$  belongs. Using the triangle inequality again, you can show that this  $\hat{x} \in \hat{X}$  really is the limit of the sequence  $(\hat{x}_n)$ .