Brachistochrone Notes

Here is one way to see the calculation through from Euler-Lagrange to the cycloid. Because the integrand $\frac{(1+\dot{x}^2)^{1/2}}{x^{1/2}}$ does not depend on t, the Euler-Lagrange equation becomes:

$$f - \dot{x}\frac{\partial f}{\partial \dot{x}} = \text{constant}$$

which in this case means that:

$$\frac{(1+\dot{x}^2)^{1/2}}{x^{1/2}} - \frac{\dot{x}}{[x(1+\dot{x}^2)]^{1/2}} = \text{constant.}$$

This simplifies to:

$$x(1+\dot{x}^2) = c\text{, constant}$$

This is an integrable first order equation for x. One way to find the solution is to introduce a new parameter z and make the substitution $\dot{x} = \tan z$. This turns our last equation into:

$$x = c(\cos^2 z)$$

which gives x as a function of the parameter z. But remember we were looking for a curve in the (t, x) plane, so we also need t as a function of z. For that, we differentiate the equation we just found for x and then use the fact that $\dot{x} = \tan z$ as follows:

$$\dot{x} = -2c\dot{z}(\cos z)(\sin z) = \tan z$$

which simplifies to:

$$1 = -2c\dot{z}\cos^2 z = -c\dot{z}(1+\cos 2z).$$

This integrates to give $-c(z + \frac{1}{2}\sin 2z) = t + \text{constant}$. So, in terms of the parameter $2z$, we conclude that the curve we want can be written as:

$$x = k(1+\cos 2z)$$
$$t = \ell - k(2z + \sin 2z)$$

where k and ℓ are constants determined by the boundary conditions. In this form, you can recognize the curve as a cycloid, that is, the trajectory followed by a point on the edge of a disc as that disc rolls along a straight line. Not also that the parameter z can be interpreted as the angle that the tangent to the curve makes with the t axis.