Mathematics 116
Convexity and Optimization with Applications

Assignment III Due in class on Monday, October 20.

Announcements Course staff will also be available on Sunday nights at 8pm at the Math Question Center in Loker Commons starting October 12.

Reading Reread Luenberger's Chapter 2 and start Chapter 3.

Exercises

1. In each of the following cases, determine the supremum of *f* over its domain *D*. If there are points *x* in *D* for which *sup f = f(x)*, find them and hence determine *max f.*

(a) $f(x) = 2x - 1,\ 0 < x < 1.$
(b) $f(x) = 3 - x^2$
(c) $f(x) = \sin(x)$
(d) $f(x) = x \sin(x)$
(e) $f(x) = 1/x$
(f) $f(x) = x$ for $0 < x < 1$ and
$1-x$ for $1 < x < 2.$
(g) $f(x) = 3x^2 - 2x - 6,\ 0 \leq x \leq 4.$

2. Consider the function space *X* which consists of all functions of a single variable of the form $f(x) = ax + b$ for all real *x*. Let *J* denote the functional on *X* defined by

$$J[f] = \int_0^1 f\,dx.$$

(a) Is every *f* in *X* a linear function on the real line according to our definition?
(b) Is *X* a vector space or what? What is its dimension? Can you find a basis? Another basis?
(c) Is *X* isometrically isomorphic to some familiar space? What does *X* need before you can answer?
(d) Evaluate $J[f]$. Is *J* linear?
(e) Can you find $sup\{ J[f] \mid f \in X \}$?
(f) What about $inf\{ J[f] \mid f \in X \}$?

3. Consider the set Y which consists of all functions of a single variable of the form $f(x) = ax$ for $0 \leq a \leq 1$. Let J denote the functional on Y defined by

$$J[f] = \int_0^a ax\,dx - a$$

for $f(x) = ax$.

(a) Is Y a subspace of the space X of the previous problem? Is J linear?
(b) Before trying to calculate $\sup \{ J[f] \mid f \in Y \}$, how do you know it must be finite and attained? Justify your answer by carefully applying an appropriate theorem.
(c) Can you find $\sup \{ J[f] \mid f \in Y \}$?
(b) What about $\inf \{ J[f] \mid f \in Y \}$?

4. Consider the sequence of functions $f_n(x) = nx(1-x)^n$ defined on $D = [0, 1]$.

(a) Show that the sequence $\{ f_n \}$ converges pointwise to some f. What is it?
(b) How do you know each f_n attains its maximum at some point x_n in D? Find them.
(c) What would you expect the sequence $\{f_n(x_n)\}$ of these maximum values should converge to? Check that the actual limit is $1/e$.
(d) Does the sequence $\{ f_n \}$ converge in C[0, 1]? I.e., does it converge uniformly?

Discussion

Come to section prepared to talk about the following questions. You can post your thoughts at the discussion section of our website (www.courses.fas.harvard.edu/~math116), preferably in advance of section, and react to other people's ideas there, too.

1. Explain in your own words what it means for a function of one variable to be continuous and relate your intuition to the formal definition. Discuss and illustrate the various kinds of discontinuities which can occur.

Challenge: Consider the function defined on the plane by setting $f(0, 0) = 0$ and

$$f(x,y) = \frac{xy}{x^2 + y^2}$$

otherwise. Is it continuous everywhere? (Hint: consider what happens along the line $x=y$ near the origin.)

2. Is the limit of continuous functions continuous? One way to understand this question is to say that it is asking whether $\lim_{x \to y} \lim_{n \to \infty} f_n(x) = \lim_{n \to \infty} \lim_{x \to y} f_n(x)$. Many questions in analysis and in its applications boil down to whether you can exchange the order of taking limits in this way. Can you think of other examples?