Euler-Lagrange Equations for Many Functions and Variables and High-Order Derivatives

The Case of High-Order Derivatives. The context is to find the extremal for the functional

$$\int_{x=a}^{b} F\left(x, y, y', \cdots, y^{(k)}\right)\, dx$$

at the function $y = f(x)$ is an extremal compared to other functions nearby, with the jets up to order $k-1$ fixed at the two end-points $x = a$ and $x = b$. In other words, $y^{(j)}(a) = A_j$ and $y^{(j)}(b) = B_j$ for $1 \leq j \leq k-1$.

We assume that we have a family of functions $y = y(x,t)$ parametrized by $t \in (-\varepsilon, \varepsilon)$ so that our assumed solution $y = y(x)$ is $y = y(x,0)$ when the parameter t is 0. Then

$$(\natural)_t \qquad \int_{x=a}^{b} F\left(x, y(x,t), \frac{\partial}{\partial x} y(x,t), \cdots, \frac{\partial^k}{\partial x^k} y(x,t)\right)\, dx$$

is a function of t and its derivative with respect to t must be equal to 0 at $t = 0$. Differentiating $(\natural)_t$ with respect to t, we get

$$\begin{aligned}
&\frac{d}{dt}\int_{x=a}^{b} F\left(x, y(x,t), \frac{\partial}{\partial x} y(x,t)\right)\, dx \\
&= \int_{x=a}^{b}\left[F_y\left(x, y(x,t), \frac{\partial}{\partial x} y(x,t), \cdots, \frac{\partial^k}{\partial x^k} y(x,t)\right)\frac{\partial}{\partial t} y(x,t)\right. \\
&\quad + F_{y'}\left(x, y(x,t), \frac{\partial}{\partial x} y(x,t), \cdots, \frac{\partial^k}{\partial x^k} y(x,t)\right)\frac{\partial^2}{\partial x \partial t} y(x,t) dx \\
&\quad + \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\
&\quad \left. + F_{y^{(k)}}\left(x, y(x,t), \frac{\partial}{\partial x} y(x,t), \cdots, \frac{\partial^k}{\partial x^k} y(x,t)\right)\frac{\partial^k}{\partial x^k \partial t} y(x,t) dx\right]
\end{aligned}$$

Setting $t = 0$, we get

$$\begin{aligned}
&\int_{x=a}^{b}\left[F_y\left(x, y, y', \cdots, y^{(k)}\right)\frac{\partial}{\partial t} y(x,0) + F_{y'}\left(x, y, y', \cdots, y^{(k)}\right)\frac{\partial^2}{\partial x \partial t} y(x,0) dx\right. \\
&\quad \left. + \cdots + F_{y^{(k)}}\left(x, y, y', \cdots, y^{(k)}\right)\frac{\partial^k}{\partial x^k \partial t} y(x,0) dx\right] = 0.
\end{aligned}$$

Integrating by parts repeatedly, we obtain

$$\int_{x=a}^{b}\Big[F_y\left(x,y,y',\cdots,y^{(k)}\right)-\frac{d}{dx}F_{y'}\left(x,y,y',\cdots,y^{(k)}\right)$$
$$+\frac{d^2}{dx^2}F_{y''}\left(x,y,y',\cdots,y^{(k)}\right)-+\cdots$$
$$+(-k)^{-1}\frac{d^k}{dx^k}F_{y^{(k)}}\left(x,y,y',\cdots,y^{(k)}\right)\Big]\frac{\partial y}{\partial t}(x,0)\,dx,$$

where all the boundary terms vanish in the process of integration by parts because the fixing of the jets up to order $k-1$ at the two end-points $x=a$ and $x=b$ implies

$$(\sharp)\qquad \frac{d^j}{dx^j}\frac{\partial y}{\partial t}(x,0)=0\ \text{ at }\ x=a\ \text{ and }\ x=b\ \text{ for }\ 1\leq j\leq k-1.$$

Since this holds for all choices of $\frac{\partial y}{\partial t}(x,0)$ for $x\in[a,b]$ as long as the condition $(\sharp)$ is satisfied, we conclude that

$$\sum_{j=0}^{k}(-1)^j\frac{d^j}{dx^j}\left(F_{y^{(j)}}\left(x,y(x),y'(x),\cdots,y^{(k)}(x)\right)\right)\equiv 0$$

on $x\in[a,b]$, which is the Euler-Lagrange equation and is in general a differential equation of order $2k$ if

$$F_{y^{(k)}y^{(k)}}\left(x,y(x),y'(x),\cdots,y^{(k)}(x)\right)$$

is nonzero (which is $(-1)^k$ times the coefficient of $y^{(2k)}(x)$).

The Case of Many Functions. The context is to find the extremal for the functional

$$\int_{x=a}^{b}F\left(x,y_1,y_1',\cdots,y_1^{(k)},\cdots,y_\ell,y_\ell',\cdots,y_\ell^{(k)}\right)\,dx$$

at the functions $y_1=f_1(x),\cdots,y_\ell=f_\ell(x)$ is an extremal compared to other functions nearby, with the jets up to order $k-1$ fixed at the two end-points $x=a$ and $x=b$. In other words, $y_\nu^{(j)}(a)=A_{j,\nu}$ and $y_\nu^{(j)}(b)=B_{j,\nu}$ for $1\leq j\leq k-1$ and $1\leq\nu\leq\ell$. We can consider the problem as the problem of variation for one single function $y_\nu=f_\nu(x)$ with all the other functions

$y_\mu = f_\mu(x)$ for $\mu \neq \nu$ fixed. Then we get for each $1 \leq \nu \leq \ell$ an Euler-Lagrange equation

$$\sum_{j=0}^{k}(-1)^j \frac{d^j}{dx^j}\left(F_{y_\nu^{(j)}}\left(x, y_1(x), y_1'(x), \cdots, y_1^{(k)}(x), \cdots, y_\ell(x), y_\ell'(x), \cdots, y_\ell^{(k)}(x)\right)\right) \equiv 0$$

on $x \in [a, b]$.

The Case of Many Independent Variables. Let $\mathbf{x} = (x_1, \cdots, x_n) \in \mathbb{R}^n$ and R be a domain in $\mathbb{R}^n$. The context is to find the extremal for the functional

$$\int_R F\left(\mathbf{x}, \{D^{\mathbf{a}}y_1, \cdots, D^{\mathbf{a}}y_\ell\}_{|a|\leq k}\right) dx_1 \cdots dx_n$$

at the functions $y_1 = f_1(\mathbf{x}), \cdots, y_\ell = f_\ell(\mathbf{x})$ is an extremal compared to other functions on R nearby, with the jets up to order $k-1$ fixed at the boundary ∂R of the given fixed domain R in $\mathbb{R}^n$, where

$$D^{\mathbf{a}} = \frac{\partial^{a_1+\cdots+a_n}}{\partial x_1^{a_1}\cdots\partial x_n^{a_n}}$$

with $\mathbf{a} = (a_1, \cdots, a_n) \in (\mathbb{N} \cup \{0\})^n$ and $|\mathbf{a}| = a_1 + \cdots + a_n$. A completely analogous derivation gives us the following Euler-Lagrange equation in this case for each $1 \leq \nu \leq \ell$

$$\sum_{|\mathbf{a}|\leq k}(-1)^{|\mathbf{a}|}D^{\mathbf{a}}\left(F_{D^{\mathbf{a}}y_\ell}\left(\mathbf{x}, \{D^{\mathbf{a}}y_1, \cdots, D^{\mathbf{a}}y_\ell\}_{|\mathbf{a}|\leq k}\right)\right) \equiv 0$$

for $\mathbf{x} \in R$.

Minimal Surface. We use the equation of a minimal surface as an example of the Euler-Lagrange equation for the case of one function, first-order derivative, and two independent variables. Let R be a bounded domain in $\mathbb{R}^2$ with variables x, y. The problem is to find the Euler-Lagrange equation for a function $z = f(x, y)$ for $(x, y) \in R$ which is a local extremal for the functional

$$\int_R \sqrt{1 + {z_x}^2 + {z_y}^2}\, dxdy$$

of the area of the graph of $z = f(x, y)$ in $\mathbb{R}^3$ over R. The Euler-Lagrange equation is

$$-\frac{\partial}{\partial x}\left(\sqrt{1 + {z_x}^2 + {z_y}^2}\right)_x - \frac{\partial}{\partial x}\left(\sqrt{1 + {z_x}^2 + {z_y}^2}\right)_y = 0,$$

which can be rewritten as

$$-\frac{\partial}{\partial x}\frac{z_x}{\sqrt{1+{z_x}^2+{z_y}^2}}-\frac{\partial}{\partial y}\frac{z_y}{\sqrt{1+{z_x}^2+{z_y}^2}}=0,$$

whose expansion is

$$-\frac{z_{xx}}{\sqrt{1+{z_x}^2+{z_y}^2}}+\frac{z_x\left(z_xz_{xx}+z_yz_{xy}\right)}{\left(1+{z_x}^2+{z_y}^2\right)^{\frac{3}{2}}}-\frac{z_{yy}}{\sqrt{1+{z_x}^2+{z_y}^2}}+\frac{z_y\left(z_xz_{xy}+z_yz_{yy}\right)}{\left(1+{z_x}^2+{z_y}^2\right)^{\frac{3}{2}}}=0.$$

After we clear the denominators by multiplying the equation by $\left(1+{z_x}^2+{z_y}^2\right)^{\frac{3}{2}}$, we get

$$z_{xx}\left(1+{z_x}^2+{z_y}^2\right)-z_x\left(z_xz_{xx}+z_yz_{xy}\right)+z_{yy}\left(1+{z_x}^2+{z_y}^2\right)-z_y\left(z_xz_{xy}+z_yz_{yy}\right)=0,$$

which can be simplified to

$$z_{xx}\left(1+{z_y}^2\right)-2z_{xy}z_xz_y+z_{yy}\left(1+{z_x}^2\right)=0.$$

This is the equation for the minimal surface.