Math 115
Fall 2006 Assignment 9 Solution Yan Zhang

Problem 1. *(#5 on p.64 of Gelfand and Fomin). Find the curves for which the functional*

$$J[y] = \int_0^{x_1} \frac{\sqrt{1+y'^2}}{y}\,dx, \quad y(0) = 0$$

can have an extrema if

(a) the point (x_1, y_1) can vary along the line $y = x - 5$;

(b) the point (x_1, y_1) can vary along the circle $(x-9)^2 + y^2 = 9$.

Answer: (a) $y = \pm\sqrt{10x - x^2}$; (b) $y = \pm\sqrt{8x - x^2}$.

Solution. The integrand is independent of x, so we can use the first integral to get $F - y'F_{y'} = c_1$ for some constant c_1, or

$$\frac{\sqrt{1+y'^2}}{y} - \frac{y'^2}{\sqrt{1+y'^2}y} = c_1.$$

Solving this, we get that $(1+y'^2)y^2 = \frac{1}{c_1^2} = c_2$ for some constant c_2. This is almost like some other problem last week, which I am too lazy to look up. Anyway, you solve this by letting $y' = \cot(\phi)$ and $y = c_2\sin(\phi)$, which eventually gives you

$$(x - c_3)^2 + y^2 = c_2$$

for some c_3. The initial conditions, however, tell you that $c_3^2 = c_2$, so you can actually cancel and get $(x-c)^2 + y^2 = c^2$ for some $c = c_3$. Note that it remains to solve for c, since this gives $y = \pm\sqrt{2cx - x^2}$.

(a) Now, we know that $g = y - x + 5$ equals 0. Using $\frac{\partial g}{\partial x}F_{y'} = \frac{\partial g}{\partial y}(F - y'F_{y'})$, we get

$$-F_{y'} = F - y'F_{y'},$$

which, when the dust clears, tells us $y'(x_1) = -1$. This means the line $y - x + 5 = 0$ intersects the circle perpindicularly. We then get $c = 5$ in our final equation. Plugging in gets $y = \pm\sqrt{10x - x^2}$, as desired.

(b) Similarly, here we do the exact same thing as above, except with $g = (x-9)^2 + y^2 - 9$. The result we get here is that $y'(x_1) = y_1/(x_1 - 9)$. solving gives $c = 4$ as the only real possibility of c, which gives us $y = \pm\sqrt{8x - x^2}$.

□

Y.Z.'s notes. A straightforward application of the rules you know. Of course, if you didn't see the x independence you can just use Euler-Lagrange - it would just be a bit more annoying. □

Problem 2. *(#10 on p.64 of Gelfand and Fomin). Find the curves for which the functional*

$$J[y,z] = \int_0^{x_1} \left(y'^2 + z'^2 + 2yz\right) dx$$

can have extrema, given that $y(0) = z(0) = 0$*, while the point* (x_1, y_1, z_1) *can vary in the plane* $x = x_1$*.*

Solution. Seeing that we have independence of x, we can use $F - y'F_{y'} = c_1$ and $F - z'F_{z'} = c_2$. For the first, we get $(y')^2 + (z')^2 + 2yz - 2(y')^2 = c_1$, or $(z')^2 + 2yz - (y')2 = c_1$. Symmetrically, we get $(y')^2 + 2yz - (z')^2 = c_2$. Adding, we get $4yz = c_1 + c_2$, so yz is a constant. Since $y(0) = z(0) = 0$, $yz = 0$ everywhere.

Now, note that where y is nonzero, z must be zero. By $(z')^2 + 2yz - (y')^2 = c_1$, we know that on places where y is nonzero, y' must be a constant, so $y = ax + b$ for some x there. It is clear that we can just have y be a line and $z = 0$, or vice-versa. But, we also have the condition that at $x = x_1$ we have $F_{y'} = 2y'$ and $F_{z'} = 2z'$ equalling 0, so the only lines that work are having $y = z = 0$. ☐

Y.Z.'s notes. Most of you got this wrong by making a move somewhere that lost a restriction. Whenever you do something like this you need to go back and use the earlier, stronger condition. These solutions are sums of trig and hyperbolic trig functions, which satisfy everything else, but if you checked the stronger conditions you would have realized that the final answer must be much more restrictive. ☐

Problem 3. *(#12 on p.65 of Gelfand and Fomin). Find the curves for which the functional*

$$J[y] = \int_0^1 \left(1 + y''^2\right) dx$$

can have extrema, given that $y(0) = 0$*,* $y'(0) = 1$*,* $y(1) = 1$*, while* $y'(1)$ *can vary arbitrarily.*

Solution. Here, we have a double derivative. Meanine we have to use (22) on page 42. This gives us $F_y - \frac{dF_{y'}}{dx} + \frac{d^2 F_{y''}}{dx^2} = 0$. The first two terms obviously die, giving the simple equation

$$2y^{''''} = 0,$$

so we know that $y = c_3x^3 + c_2x^2 + c_1x^1 + c_0$. The conditions of the problem give that $c_0 = 0$, $c_1 = 1$, $c_3 + c_2 + c_1 + c_0 = 1$ respectively. So we know that $c_3 = -c_2$, meaning the answer is

$$y = cx^3 - cx^2 + x.$$

☐

Problem 4. *(#15 on p.65 of Gelfand and Fomin). Prove that the functional*

$$J[y] = \int_{x_0}^{x_1} \left(ay'^2 + byy' + cy^2\right) dx, \quad y(x_0) = y_0, \quad y(x_1) = y_1,$$

where $a \neq 0$*, can have no broken extremals.*

Solution. Take any point t. The Weierstrass-Erdmann conditions tell us that $F_{y'}|_{t^-} = F_{y'}|_{t^+}$ and $(F - y'F_{y'})|_{t^-} = (F - y'F_{y'})|_{t^+}$. The first condition translates to

$$2ay' + by|_{t^-} = 2ay' + by|_{t^+}.$$

Now, y must be continuous, so we can ignore the second term. But this means $2ay'$ is continuous. Here we use the condition that $a \neq 0$, so y' must be continuous as well. Thus, we have no broken extremals. ☐

Y.Z.'s notes. A fun mathematical exercise is to generalize this - what classes of such functionals only admit non-broken extremals? ☐

Problem 5. *(#21 on p.66 of Gelfand and Fomin). Find the curves for which the functional*

$$J[y] = \int_0^{10} {y'}^3 dx, \quad y(0) = 0, \quad y(10) = 0$$

can have extrema, given that the admissible curves cannot penetrate the interior of the circle with equation

$$(x-5)^2 + y^2 = 9.$$

Answer:

$$y = \begin{cases} \pm\frac{3}{4}x & \text{for} \quad 0 \leq x \leq \frac{16}{5}, \\ \pm\sqrt{9-(x-5)^2} & \text{for} \quad \frac{16}{5} \leq x \leq \frac{34}{5}, \\ \mp\frac{3}{4}(x-10) & \text{for} \quad \frac{34}{5} \leq x \leq 10. \end{cases}$$

Solution. I'm going to only sketch this proof, since most of you guys got the right idea and being more rigorous would not be very productive.

First, you need to realize that any segment off of the circle must be a straight line. your standard analysis of the first-derivative test tells you this. This means that either the final solution is a straight line, which is absurd, or that our solution consists of three parts: a line to the circle, an arc of the circle, nad a line off of the circle.

Now, by using Weierstrass-Erdmann conditions on where the line meets the circle, we can find that the line must be tangent to the circle (which makes intuitional sense). So that is the answer given. ☐

Y.Z.'s notes. Theoretically we should have been careful near the boundary of the circle to use the Weierstrass-Erdmann conditions, but it seems it is obvious we have freedom there. I'm not too sure how to make this part rigorous myself, so you are all excused from it. ☐