Math 115
Fall 2006 Assignment 8 Solution Yan Zhang

Problem 1. *(Poisson's Integral Formula for the Exterior of a Circle [#4 on p.167 of Strauss]). Suppose* u *is a bounded twice continuously differentiable function on the exterior* $\{\, x^2 + y^2 > a^2 \,\}$ *of the circle* C_a *of radius* $a > 0$ *centered at the origin. Assume that* u *is continuous up to the circle* C_a *and let* $h(\theta) = u\left(e^{i\theta}\right)$ *for* $0 \le \theta \le 2\pi$*. Derive the following Poisson's integral formula for the exterior of the circle* C_a*.*

$$u\left(re^{i\theta}\right) = \left(r^2 - a^2\right) \int_{\varphi=0}^{2\pi} \frac{h\left(\varphi\right)}{a^2 - 2ar\cos\left(\theta - \varphi\right) + r^2} \frac{d\varphi}{2\pi}$$

for $r > a$*.*

Hint: Imitate the argument used in the derivation of the Poisson integral formula by Fourier series and convolution.

Solution. Use $z \Rightarrow a/z$ to send the domain outside of the circle with radius a to the interior of the unit disc. Now our function is twice the real part of some holomorphic function $f(a/z)$ (Our function was bounded (i.e. we do not have poles outside, including at infinity), so a/z gives a holomorphic function).

$$\begin{aligned}
u(re^{i\theta}) &= 2Re(f(a/z)) \qquad (1)\\
&= c_0 + \overline{c_0} + \sum_{n=1}^{\infty} c_n (a/r)^n e^{-in\theta} + \sum_{n=1}^{\infty} \overline{c_n} (a/r)^n e^{in\theta} \qquad (2)\\
&= (c_0 + \overline{c_0} + \sum_{n=1}^{\infty} c_n e^{-in\theta} + \sum_{n=1}^{\infty} \overline{c_n} e^{in\theta}) * (1 + \sum_{n=1}^{\infty} (a/r)^n e^{-in\theta} + \sum_{n=1}^{\infty} (a/r)^n e^{in\theta}) \qquad (3)\\
&= u(e^{i\theta}) * Re(\frac{1 + (a/r)e^{i\theta}}{1 - (a/r)e^{i\theta}}) \qquad (4)\\
&= u(e^{i\theta}) * [\frac{a^2 - r^2}{a^2 + r^2 - 2ar\cos(\theta)}]. \qquad (5)
\end{aligned}$$

Now we are done by the definition of convolution.

☐

Y.Z.'s notes. I hope you are beginning to get the big idea of why conformal maps are useful. Note that between the last set and this set, we did quite a lot on different domains (being outside of a disc, upper-half plane, etc.) by just conformally mapping what we know inside the unit disc. I hope these examples have been instructive for this big picture of problem solving - to have the problem solved in a specific case and then generalize as much as possible. ☐

Problem 2. *(Uniqueness of Harmonic Solution with the Robin Condition [#11 on p.168 of Strauss]). The boundary condition for solving the Laplace equation* u *on a bounded domain* D *is called the Dirichlet condition if the value of* u *is specified at the boundary* ∂D *of* D*. The boundary condition for solving the Laplace equation* u *on a bounded domain* D *is called* *the Neumann condition if the outward normal derivative* $\frac{\partial u}{\partial \mathbf{n}}$ *of* u *is specified at the boundary* ∂D *of* D. *The boundary condition for solving the Laplace equation* u *on a bounded domain* D *is called the Robin condition if the normal derivative* $\frac{\partial u}{\partial \mathbf{n}} + au$ *is specified at the boundary* ∂D *of* D, *where* a *is a function on the boundary* ∂D *of* D.

Prove the uniqueness of the Robin problem

$$\Delta u = 0 \text{ in } D, \quad \frac{\partial u}{\partial \mathbf{n}} + au = 0 \text{ on } \partial D,$$

where D *is any domain in three dimensions and where* a *is a positive constant.*

Hint: Imitate the argument used in the proof of the uniqueness for the Neumann boundary value problem from the divergence theorem.

Solution. We know that

$$\int_{\partial D} u\frac{\partial u}{\partial \vec{n}} + au^2 = 0$$

since the integrand is 0. By integration by parts like we did in lecture, this is

$$\int_D (|grad(u)|^2 + u\Delta u) + \int_{\partial D} au^2 = \int_D |grad(u)|^2 + \int_{\partial D} au^2.$$

We know that $a > 0$, so the only way for this to be 0 is to have both integrands being 0. So u is 0 on the boundary and has 0 gradient inside, meaning u must actually be 0 everywhere. □

Problem 3. *(Hopf Lemma [#12 on p.168 of Strauss]). (a) Let* $0 < a < b$. *Let* $G_{a,b}$ *be the domain defined by* $a < r < b$, *where* $r = |\mathbf{x}|$ *and* $\mathbf{x}$ *is a point in* $\mathbb{R}^2$ *(or in general in* $\mathbb{R}^n$*). Let* $\sigma > 0$ *and let* $v_\sigma(\mathbf{x}) = e^{-\sigma r^2} - e^{-\sigma b^2}$. *Verify that there exists some* $\sigma_0 > 0$ *such that for* $\sigma \geq \sigma_0$ *one has* $\Delta v_\sigma > 0$ *on* $\overline{G_{a,b}}$, *where* Δv_σ *is the Laplacian of* v_σ *and* $\overline{G_{a,b}}$ *is the closure of* $G_{a,b}$. *Verify also that* $\frac{\partial}{\partial \mathbf{n}} v_\sigma < 0$ *at every point of* $r = b$, *where* $\frac{\partial}{\partial \mathbf{n}}$ *means differentiation in the direction of the outward normal* $\mathbf{n}$ *of* $G_{a,b}$.

(b) Suppose w *is a twice continuous differentiable real-valued function on some open neighborhood of the origin such that it assumes its local maximum at the origin. Verify that the Laplacian of* w *at the origin must be* ≤ 0.

Hint: Use the second derivative test.

(c) Prove the following strong form of the maximum principle, called the Hopf form of the maximum principle. If $u(\mathbf{x})$ *is a nonconstant harmonic function on a domain* D *that has a maximum at* $\mathbf{x}_0$ *(necessarily on the boundary* ∂D *of* D*), then* $\frac{\partial u}{\partial \mathbf{n}} > 0$ *at* $\mathbf{x}_0$, *where* $\frac{\partial u}{\partial \mathbf{n}}$ *is the outward normal derivative of* u.

Hint: Without loss of generality we can assume that $G_{a,b}$ *from Part (a) is contained in* D *so that* $\{r = b\}$ *touches* ∂D *at* $\mathbf{x}_0$. *Verify that there exists some* $\varepsilon_0 > 0$ *such that for* $0 < \varepsilon < \varepsilon_0$ *and for* $\sigma \geq \sigma_0$ *one has* *(i)* $\Delta\left(u - u\left(\mathbf{x}_0\right) + \varepsilon v_\sigma\right) > 0$ *on* $G_{a,b}$*,*

(ii) $u - u\left(\mathbf{x}_0\right) + \varepsilon v_\sigma \leq 0$ *on* $\partial G_{a,b} = \{r = a\} \cup \{r = b\}$ *and hence also on all of* $G_{a,b}$ *by (b) and (i),*

(iii) $\frac{\partial u}{\partial \mathbf{n}} \geq -\varepsilon \frac{\partial v_\sigma}{\partial \mathbf{n}}$ *at* $\mathbf{x}_0$ *from (ii).*

Solution. I was going to type up a solution, but a fairly clear proof is given at

http://ocw.mit.edu/NR/rdonlyres/Mathematics/18-152Fall-2005/3A85692B-D5D0-4A45-8993-3A31F9C955AE/0/lecture3.pdf

using the same sort of argument as the hint. Note that they also give a bound, which is nice. If you don't care about the bound, you can ignore the last page.

□

Problem 4. *Let* $-\infty < x_1 < x_2 < \infty$ *and* $y_1, y_2 \in \mathbb{R}$*. Find the extremals* $y = y(x)$ *for the following functionals with* $y\left(x_1\right) = y_1$ *and* $y\left(x_2\right) = y_2$*.*

(a)
$$\int_{x_1}^{x_2} \sqrt{y\left(1 + y'^2\right)}\, dx.$$

(b)
$$\int_{x_1}^{x_2} \frac{1 + y^2}{y'^2}\, dx.$$

(c)
$$\int_{x_1}^{x_2} y'\left(1 + x^2 y'\right) dx.$$

(d)
$$\int_{x_1}^{x_2} \left(y^2 + y'^2 - 2y \sin x\right) dx.$$

Solution. (a) Without dependence on x, we can juse use the first integral, or $F - y'F_{y'} = c_0$. Solving, we get

$$c_1 = \frac{y}{1 + y'^2}.$$

This should remind you of the tangent. Let $y' = \tan(\theta)$. Then $y = c_1 \sec^2(\theta)$, and $dy = 2c_1 \sec^2(\theta)\tan(\theta)d\theta$, $dx = \frac{dy}{\tan(\theta)=} 2c_1 \sec^2(\theta)d\theta$. Integrating both, we get that $x = 2c_1 \tan(\theta) + c_2$, $y = 2c_1 \tan^2(\theta) + c_3$, so our curve is

$$y = \frac{(x - c_3)^2}{2c_1} + c_3,$$

where we can obviously solve the constants.

(b) Similar to the last one, we can use $F - y'F_{y'} = c_0 = \frac{1+y^2}{y'^2}$. This gives the relation that

$$\frac{dy}{\sqrt{1+y^2}} = c_1 dx.$$

Integrating gives $\sinh^{-1}(y) = c_1 x + c_2$, or

$$y = \sinh(c_1 x + c_2).$$

(c) We can no longer do the same trick as the first two. However, note we have no y dependence, so $F_y = 0$. Thus, because $F_y = \frac{d}{dx}F_{y'}$, we have

$$\begin{aligned} 0 &= \frac{d}{dx}(1 + 2x^2 y') && (6)\\ 0 &= 4xy' + 2x^2 y'' && (7)\\ -\frac{2}{x} &= \frac{y''}{y'} && (8)\\ \log(x^{-2} + c_0 &= \log(y') && (9)\\ \frac{c_1}{x^2} &= y', && (10) \end{aligned}$$

which gives us

$$y = \frac{-c_1}{x^2} + c_2.$$

(d) Now we get no frebies, so we just do it with $F_y = \frac{d}{dx}F_{y'}$:

$$\begin{aligned} 2y - 2\sin(x) &= 2y'' && (11)\\ y'' - y + \sin(x) &= 0. && (12) \end{aligned}$$

This is pretty standard, you know the answer is $c_0 e^{-x} + c_1 e^x + f(x)$ since the polynomial $(x^2 - 1)$ has the roots 1 and -1. We then just need an $f(x)$ which satisfies the above equation. $\sin(x)/2$ happens to work, so the final answer is

$$y = \sin(x)/2 + c_0 e^{-x} + c_1 e^x.$$

□