Problem 1. *(Poisson's Integral Formula for the Upper Half-Plane [#1 on p.171 of Ahlfors]). Verify Part (a) by imitating the derivation of the Poisson integral formula for the unit disk by the method of the argument function and plane Euclidean geometry and by using an $\mathbb{R}$-linear combination of the constant function* 1 *and the two functions*

$$\arg(z-\alpha), \quad \arg(z-\beta)$$

for $-\infty < \alpha < \beta < \infty$ and passing to the limit of its quotient by $\beta - \alpha$ as $\beta \to \alpha$.

(a) Assume that $U(\xi)$ is piecewise continuous and bounded for all real ξ. Show that

$$P_U(z) = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{y}{(x-\xi)^2+y^2}\,U(\xi)\,d\xi$$

represents a harmonic function in the upper half-plane with boundary value $U(\xi)$ at points of continuity.

(b) Verify Part (a) by using the Poisson integral formula for the unit disk and a linear fractional transformation which maps conformally the unit disk onto the upper half-plane.

Solution. (a) We know our measure is going to be

$$\omega_{ab} = \frac{1}{\pi}(arg(z-b) - arg(z-a)) \quad (1)$$
$$\omega_{ab} = \frac{1}{\pi}(tan^{-1}\frac{y}{x-b} - tan^{-1}\frac{y}{z-a}) \quad (2)$$
$$(3)$$

where we are defining $arg(z) = tan^{-1}\frac{Im(z)}{Re(z)}$, with the branch for the angle and the arctan function being the branch between 0 and π (since we are in the upper half plane). We want the limit

$$\lim_{b \Rightarrow a} \frac{\omega_{ab}}{b-a} = \frac{y}{(x-b)^2+y^2}.$$

You can get this by using L'Hopital's rule if you are inclined to do so. Another way is to draw some right triangles with Euclidiean geometry. In any case, the result follows pretty immediately. To see it is harmonic, you can differentiate under the integral twice respect to x an y. I'll skip the easy algebra.

(b) Here, notice that the conformal map $z \Rightarrow i\frac{1+z}{1-z}$ sends the unit disk to the upper half plane with inverse $z \Rightarrow \frac{z-i}{z+i}$. Then the term in the Poisson integral formula becomes

$$\frac{1-|z|^2}{|z-\zeta|^2} = \frac{1-|\frac{z-i}{z+i}|^2}{|\frac{z-i}{z+i}-\frac{\zeta-i}{\zeta+i}|^2} \tag{4}$$

$$= \frac{\frac{4y}{|z+i|^2}}{\frac{4(x-\zeta)^2+y^2}{|z+i|^2|\zeta+i|^2}} \tag{5}$$

$$= \frac{y}{\frac{(x-\zeta)^2+y^2}{\zeta^2+1}} \tag{6}$$

$$= \frac{y(\zeta^2+1)}{(x-\zeta)^2+y^2}. \tag{7}$$

So when we put it into the Poisson integral formula, $\zeta^2+1=2$, canceling out the 2 in the $1/(2\pi)$ factor, and we get exactly what we wanted.

□

Y.Z.'s notes. Part (a) can also be done (and should be done) with a geometric argument. I just don't like drawing pictures in LaTeX. Also, by doing part (b) you should have the nagging suspicion that this is just doing what we have already done in class in a different "coordinate system." Indeed, if you went back and go through the original poisson's formula and use the map $z \Rightarrow i\frac{1+z}{1-z}$ everywhere, every step can be translated exactly to a step in part (a). This is another way to get a geometric argument. □

Problem 2. *(Pointwise Convergence of the Cesàro Sum of the Fourier Series of a Continuous Periodic Function). Let $f(x)$ be a continuous function on $\mathbb{R}$ with period 2π. Let s_n be the n-th partial sum of the Fourier series of $f(x)$ (in terms of the cosine and sine functions) so that*

$$s_n = \frac{a_0}{2} + \sum_{k=1}^{n}\left(a_k \cos kx + b_k \sin kx\right),$$

where

$$a_k = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos kx\,dx,$$
$$b_k = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin kx\,dx$$

are by definition the Fourier coefficients of $f(x)$. Let σ_n denote the n-th Cesàro sum of the Fourier series of $f(x)$ defined by

$$\sigma_n = \frac{s_0+s_1+\cdots+s_{n-1}}{n}.$$

Let $D_n(x)$ denote the n-th Dirichlet-Dini kernel which is defined by

$$D_n(x) = \frac{1}{2\pi}\sum_{k=-n}^{n} e^{ikx} = \frac{1}{2\pi}\frac{\sin\left(n+\frac{1}{2}\right)x}{\sin\frac{x}{2}}.$$

Define the Féjer kernel $F_n(x)$ by

$$F_n(x) = \frac{D_0(x) + D_1(x) + \cdots + D_{n-1}(x)}{n}.$$

(a) Verify that $\sigma_n = f * F_n$. Here the notation $f * F_n$ denotes the convolution of the two functions $f(x)$ and $F_n(x)$ defined by

$$\left(f * F_n\right)(x) = \int_{y=-\pi}^{\pi} f(x-y) F_n(y) dy.$$

(b) Verify that

$$F_n(x) = \frac{1}{2\pi n} \frac{\sin^2 \frac{nx}{2}}{\sin^2 \frac{x}{2}}.$$

(c) Verify that the family of functions $\{F_n(x)\}_{n\in\mathbb{N}}$ on $\mathbb{R}$ with period 2π is an approximate identity on $[-\pi, \pi]$ in the sense that the following three conditions are satisfied.

(i) (Nonnegativity) $F_n(x) \geq 0$ for $x \in [-\pi, \pi]$ and $n \in \mathbb{N}$.

(ii) (Unit Integral) $\int_{-\pi}^{\pi} F_n(x)\, dx = 1$ for all $n \in \mathbb{N}$.

(iii) (Integral Outside Any Neighborhood of the Origin Approaching 0) For any $\eta > 0$ the integral

$$\int_{\substack{-\pi \leq x \leq \pi \\ |x| \geq \eta}} F_n(x)\, dx$$

approaches 0 as $n \to \infty$.

(d) Use Part (c) to show that $\lim_{n\to\infty} \sigma_n(x) = f(x)$. This kind of convergence of the average of partial sums is known as the convergence of the Cesàro sum or convergence in the sense of Cesàro.

Solution. (a) Knowing the commutativity of convolution, we proceed thus:

$$\begin{aligned}
(f * F_n)(x) &= (F_n * f)(x) && (8)\\
&= \int_{-\pi}^{\pi} f(y) F_n(x-y) dy && (9)\\
&= \frac{1}{n}\int_{-\pi}^{\pi} f(y)(D_0(x-y) + D_1(x-y) + \ldots + D_{n-1}(x-y))dy && (10)\\
&= \frac{1}{2\pi n}\int_{-\pi}^{\pi} f(y)(\sum_{k=0}^{0} e^{ik(x-y)} + \sum_{k=-1}^{1} e^{ik(x-y)} + \ldots + \sum_{k=-n-1}^{n-1} e^{ik(x-y)})dy && (11)\\
&= \frac{1}{2\pi n}\int_{-\pi}^{\pi} f(y)((n-1) + (n-2)2\cos(x-y) + \ldots + 2\cos{(n-1)(x-y)})dy, && (12)
\end{aligned}$$

where the last line comes from summing how many pairs the sum $e^{i(x-y)} + e^{-i(x-y)}$ appears in the summation. Now it is a matter of bookkeeping - note that each $\cos(kx - ky) = \cos(kx)\cos(ky) + \sin(kx)\sin(ky)$, so when we integrate we get $a_k \cos(kx) + b_k \sin(kx)$. Meaning when we do the whole sum we eventually get

$$\frac{1}{n}((n-1)\frac{a_0}{2} + (n-2)(a_1\cos(x) + b_1\sin(x)) + \ldots + (1)(a_{n-1}\cos((n-1)x) + b_{n-1}\sin((n-1)x)))$$

or

$$\frac{1}{n}(\sigma_0 + \sigma_1 + \ldots + \sigma_{n-1}),$$

as desired.

(b) Clearly, we can write

$$\begin{aligned}
F_n(x) &= \frac{1}{2\pi n}\sum_{k=0}^{n-1}\frac{\sin((k+1/2)x)}{\sin(x/2)} && (13)\\
&= \frac{1}{2\pi n\sin^2(x/2)}\sum(\sin(x/2)\sin((k+1/2)x)) && (14)\\
&= \frac{1}{4\pi n\sin^2(x/2)}\sum(\cos(kx) - \cos((k+1)x)) && (15)\\
&= \frac{1}{4\pi n\sin^2(x/2)}(1 - \cos(nx)) && (16)\\
&= \frac{\sin^2(nx/2)}{2\pi n\sin^2(x/2)}. && (17)
\end{aligned}$$

(c)

(i) The first part is trivial by what we just did in (b).

(ii) For the second part, note that when we integrate each D_k, the constant gets integrated but everything else cancels (since they come in pairs, meaning they are actually some cosine, which integrates to 0 on $(-\pi, \pi)$). We pick up a copy of 2π from the integration and one constant of 1 for each summand, so we get 1.

There is a somewhat tricky way to do a part of (c) that some of you came up with. This involves first showing that $\int_0^\pi \frac{\sin(2n+1)x}{\sin(x)}dx = \pi$, then $\int_0^\pi \frac{\sin^2(nx)}{\sin^2(x)}dx = n\pi$. Both are done by induction, which is kind of unexpected. This is more complex than necessary, but there is good practice in doing these integrals, which ironically have come up in prelim math grad school questions, as I've found out.

The final, really slick way of doing this is to recognize the integral as $(1 * F_n)(x)$. Now use part (a). The fourier decomposition of 1 is very easy. $a_0/2 = 1$, and all other terms are 0, so $\sigma_n = 1$ and we're done.

(iii) There were some BS-like ways of approaching this. But it is really enough just to use your result from part (b). The numerator is bounded above by 1, and the denominator is bounded below by a constant away from 0. Thus, the whole thing is bounded by a constant over n, which goes to ∞.

(d) Part (c) tells us that $\lim_{n\Rightarrow\infty} F_n(x) = \delta(x)$. So when we take the limit of σ_n, we get $(f * \delta)(x) = f(x)$, since δ is the identity under the convolution operator.

$\square$

Y.Z.'s notes. Hmm, I wonder if there is an easier way to do (b) coming from (a)? Otherwise, you should read the different ways to do the second part of (c) since they are all instructive. The final thing you should get away from this is thinking of the delta function as the identity in convolution. You should meet convolution many more times in random contexts. $\square$

Problem 3. *(Solution of the Laplace Equation on the Unit Disk with Neumann Boundary Condition). For a complex variable ζ and another complex variable z with $|z| < |\zeta|$, the gradient of $\log|z-\zeta|$ with respect to z is equal to the negative of the gradient of $\log|z-\zeta|$ with respect to ζ. Moreover, by the Cauchy-Riemann equations for the variable ζ, the outward radial directive of $\log|z-\zeta|$ with respect to ζ is equal to the tangential derivative of a suitably defined branch of $\arg(z-\zeta)$ along the circle of radius $|\zeta|$ centered at the origin in the counterclockwise sense. On the other hand, for $|\zeta| = 1$ the tangential derivative of $\arg(z-\zeta)$ with respect to ζ along the unit circle is related to*

$$\lim_{\beta\to\alpha} \frac{1}{\beta-\alpha} \arg \frac{z-e^{i\beta}}{z-e^{i\alpha}},$$

which, in turn, is related to the Poisson integral kernel constructed by the method of the argument function and plane Euclidean geometry. This observation motivates the following way of constructing a solution of the Laplace equation on the unit disk with Neumann boundary condition.

For z inside the unit disk and for ζ on the unit circle, let

$$Q(z,\zeta) = -2\log|z-\zeta|.$$

Let $g(\zeta)$ be a continuous function on the unit circle $|\zeta| = 1$ such that

$$(*) \qquad \int_{\varphi=-\pi}^{\pi} g\left(e^{i\varphi}\right) d\varphi = 0.$$

For $z \in \mathbb{C}$ with $|z| < 1$, let

$$u(z) = \frac{1}{2\pi}\int_{\varphi=-\pi}^{\pi} Q\left(z, e^{i\varphi}\right) g\left(e^{i\varphi}\right) d\varphi.$$

Verify that $u(z)$ is harmonic on the open unit disk $\{|z| < 1\}$ and that for every fixed $\varphi \in \mathbb{R}$ the radial derivative

$$\frac{\partial}{\partial r} u\left(re^{i\varphi}\right)$$

of u approaches $g\left(e^{i\varphi}\right)$ as $r \to 1$ from $r < 1$.

Note that the function $g(\zeta)$ which defines the Neumann boundary condition must satisfy $(*)$ because of the divergence theorem for the gradient of the harmonic function

Solution. Note the only part depending on x and y inside the integral is $Q = -2\log(|x + iy - \zeta|)$. $\frac{\partial^2}{partial x^2} Q = \frac{-2}{|x+iy-\zeta|^2}$. When we do the same to y, since it has an i coefficient, we will pick up an extra $i^2 = -1$, so we have $\Delta Q = 0$ and Q is harmonic.

Note that one can also see this by realizing that $Q = -2\log(|x + iy - \zeta|)$ is the real part of $F = -2\log(x + iy - \zeta)$, which is holomorphic. Continuing off of this idea, define $P = -2arg(z - \zeta)$ to be the imaginary part of F.

Now we have, by the polar version of the Cauchy-Riemann equations,

$$\frac{\partial Q}{\partial r} = \frac{1}{r}\frac{\partial P}{\partial \theta} \tag{18}$$

$$= \lim_{\epsilon \Rightarrow 0^+} \frac{2}{r}\frac{arg(re^{i\theta} - e^{i\phi}) - arg(re^{i\theta+\epsilon} - e^{i\phi})}{\epsilon} \tag{19}$$

$$= \frac{K}{r}, \tag{20}$$

where K is the Poisson kernel. As $r \Rightarrow 1$, this approaches K, so the integral goes to $g(e^{i\theta})$ since the Poisson kernel picks up the value at $e^{i\theta}$.

□