Math 115
Fall 2006 | Assignment 5 Solution | Yan Zhang

Problem 1. *Let T denote the bounded steady temperature on the first quadrant $\{x > 0, y > 0\}$ with the following three constraints:*

(i) The boundary value of T on $\{x > 1, y = 0\}$ is 1.

(ii) The boundary value of T on $\{x = 0, y > 1\}$ is 0.

(iii) Both the boundary-segment $\{x = 0, 0 < y < 1\}$ and the boundary-segment $\{0 < x < 1, y = 0\}$ are insulated.

Verify that

$$T = \frac{1}{2} + \frac{1}{\pi}\sin^{-1}\frac{1}{2}\left(\sqrt{\left(x^2-y^2+1\right)^2+4x^2y^2} - \sqrt{\left(x^2-y^2-1\right)^2+4x^2y^2}\right),$$

where the range of the inverse sine function is chosen to be $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$.

Solution. First, the map $z_1 = z^2$ sends our region to the upper-half-plane and the insulated portion to the segment $(-1, 1)$ on the x-axis. Then, $z_2 = \sin^{-1}(z_1 0)$ sends this region perfectly to the infinite rectangle bounded by the segment $(-\pi/2, \pi/2)$ on the x-axis and the two vertical lines $x = \pm\pi/2$. Solving for the coefficients to make the desired values 1 and 0 on the two vertical lines, we get

$$T = 1/2 + (1/\pi)Re[\sin^{-1}(z^2)].$$

Using the formula

$$Re[\sin^{-1}(x+iy) = \sin^{-1}[(1/2)(\sqrt{(x+1)^2+y^2} - \sqrt{(x-1)^2+y^2})],$$

some algebra gets our answer. Remember that at somepoint we are making a branch cut for the arcsine, as well as for the two square roots. The obvious ones work. □

Y.Z.'s notes. This is straightforward exercise of the techniques you have learned so far. □

Problem 2. *Let $H > 0$ and $A > 0$. Let D be the domain obtained from the upper half-plane $\{y > 0\}$ by removing the line-segment $\{x = 0, 0 < y \leq H\}$. Consider the 2-dimensional incompressible irrotational fluid flow in D whose velocity at infinity is horizontal from left to right with speed A. Find the velocity of the flow at the point $(x, y) \in D$.*

(The practical interpretation of the problem is that there is a dam of height H and the flow is infinitely deep with velocity A at infinity.)

Solution. The sequence of maps

$$z_1 = z^2 \quad (1)$$
$$z_2 = z_1 + H^2 \quad (2)$$
$$z_3 = \sqrt{z^2} \quad (3)$$

gives us a picture of the upper half plane with uniform flow A from left to right. In this picutre, $F = Az_3 = A\sqrt{z^2 + H^2}$. Thus, we know that our velocity is

$$v = \overline{F'} = \overline{\frac{2Az}{\sqrt{z^2 + H^2}}},$$

which we can easily write in terms of x and y if desired.

□

Y.Z.'s notes. There is another approach - note that if you take the picture given in the problem and flip it across the x-axis and superimpose, you get a fluid flow system that, when restricted to the upper-half-plane, has the same flow as our original problem at every point by symmetry thanks to the fact that our fluid is not viscous and no flow goes through the x-axis. Then doing the problem in a similar fashion to the notes (since we have done a problem like this already) will also yield an answer - and it must be correct and equivalent to ours, since as we know our answer must be unique. □

Problem 3. *Let D be the domain obtained from $\{0 < y < \pi\}$ by removing the line-segment joining the origin to $\left(0, \frac{\pi}{2}\right)$. Find an orientation-preserving conformal mapping F which maps D to the upper half-plane.*

Hint: First apply the map $z \mapsto e^z$. Then apply the map $z \mapsto \frac{z-1}{z+1}$.

Solution. Following Prof. Siu's hint, the first map sends our picture to the upper-half plane minus an arc of the unit circle clockwise from 1 to i. The second map sends this to the upper-half-plane minus the segment from 0 to i. Note we get exactly the picture we had in Problem 2!

Now, we may repeat our argument in problem 2 for $H = 1$. This gives us

$$\omega = \sqrt{(\frac{e^z - 1}{e^z + 1})^2 + 1}.$$

□

Y.Z.'s notes. If you did not have the hint, one way to possibly guess that you'd want an exponential map is a line of reasoning like this: the log map tends to give you a horizonal strip in the plane, and that is basically what you have in the beginning besides the little segment. Therefore, to smooth it out, you might want to do the inverse of the log map first, i.e. the exponential map. □

Problem 4. *Let D be as in Problem 3. Consider the 2-dimensional incompressible irrotational fluid flow in D whose velocity at infinity is horizontal from left to right with speed A. Use the result in Problem 3 to find the velocity of the flow at the point $(x, y) \in D$.*

(The practical interpretation of the problem is for a flow in a very wide channel of height π with an obstructive wall of height $\frac{\pi}{2}$.)

Solution. After we do the transform in #3, we get

$$\omega = \sqrt{(\frac{e^z - 1}{e^z + 1})^2 + 1}.$$

And in the plane for ω, we have a source eminating from $(-\sqrt{2}, 0)$ and a sink going into $(\sqrt{2}, 0)$. We do these two parts by superposition. Suppose we just had the source. Then the map $\omega_1 = \omega + \sqrt{(2)}$ moves the source to the origin, and taking $\omega_2 = \log(\omega_1)$ and taking the branch to be 0 to π gives a strip with uniform flow from left to right, so we get

$$F_1 = A\omega_2 = A\log(w + \sqrt{2}).$$

doing a similar calculation for the sink, we get

$$F_2 = A\omega_2 = -A\log(w - \sqrt{2}).$$

Thus, by superposition our final answer should be the sum of the two, or

$$F = A\omega_2 = A\log(\frac{w + \sqrt{2}}{w - \sqrt{2}}).$$

Now, to evaluate v, we just differentiate F and take the conjugate to get the velocity vector, as we have done previously.

□

Y.Z.'s notes. Many of you did this calculation for the source but not the sink. The problem's symmetric nature should warn you that you were only halfway done. The important thing to remember from this exercise is that when we do conformal mappings, follow *everything* that happens to the boundary, especially at infinity. □

Problem 5. *Let D be the domain obtained from $\mathbb{C}$ by removing the two line segments $\{x = 0, 1 \leq y < \infty\}$ and $\{x = 0, -\infty < y \leq -1\}$. Consider a flow in D from left to right whose velocity at infinity is horizontal. Assume that the net rate of the flow across $\{x = 0, -1 < y < 1\}$ is A. Find the velocity of the flow at the point $(x, y) \in D$.*

Hint: The net rate A of the flow across $\{x = 0, -1 < y < 1\}$ is equal to the difference between the constant value of the streamline function on $\{x = 0, 1 \leq y < \infty\}$ and the constant value of the streamline function on $\{x = 0, -\infty < y \leq -1\}$, because A is the integral over $\{x = 0, -1 < y < 1\}$ of the horizontal component of the flow velocity and one can apply the Cauchy-Riemann equation and the Fundamental Theorem of Calculus.

Solution. For the net rate of flow to be A, the information we get is taht we can assign the boundary values for ψ to be $A/2$ and $-A/2$ at the two vertical rays (or anything else with difference A - it should not matter). It remains to do some conformal mappings.

The map $z_1 = iz$ rotates the picture, and $z_2 = \sin^{-1}(z_1)$ gives us the vertical infinite strip bounded between $x = -\pi/2$ and $x = \pi/2$, where we assign the ψ to be $-A/2$ and $A/2$ respectively. Solving for these boundary conditions, we find that

$$F = (A/\pi)(z_2) = (A/\pi)\sin^{-1}(iz).$$

The rest is what we have done many times - take a derivative and then take a conjugate. □

Y.Z.'s notes. The hard part for this problem is really just knowing what the description for flow really means. This is what the hint about the fundamental theorem of calculus is for. □