Math 115
Fall 2006 Assignment 4 Solution Yan Zhang

Problem 1. *(a) Verify the following infinite partial fraction decomposition*

$$\frac{1}{e^z-1} = \frac{1}{z} - \frac{1}{2} + 2z\sum_{n=1}^{\infty} \frac{1}{z^2+4n^2\pi^2}.$$

(b) Verify the following infinite partial fraction decomposition

$$\tan z = 2z\sum_{n=0}^{\infty} \frac{1}{\left(n+\frac{1}{2}\right)^2\pi^2 - z^2}.$$

(c) Verify the following infinite product expansion

$$\cos z = \prod_{n=1}^{\infty}\left(1 - \frac{z^2}{\left(n-\frac{1}{2}\right)^2\pi^2}\right).$$

Solution. (a) First note that $1/(e^z-1)$ has poles at 0 and $2n\pi i$ for $n \in \mathbf{Z}$. By L'Hopital or whatever, realize that $1/(e^z-1) - 1/z$ has no poles at 0, and has value $-1/2$.

Now, you need to show that you can draw a family of boundaries C_n with distances R_n from the origin, and get a p such that $|f(z)/R_n^{p+1}|$ is uniformly bounded by something going to 0. Then you are allowed to write

$$f(z) = \sum_0^p z^i f^{(i)}(0)/i! + \sum_1^\infty b_n(1/(z-a_n) + 1/a_n + z^2/a_n^2 + \ldots + z^p/a_n^{p+1}).$$

In our case, your standard dumb boundary works (just draw a big rectangle), and $p = 0$ actually works. So we have

$$1/(e^z-1) - 1/z = -1/2 + \sum_{-\infty}^{\infty}(1/(z-2\pi ni) + 1/(2\pi ni)),$$

which gets exactly what we want when you group them in pairs.

(b) This problem has no new tricks like above - the same reasoning works.

(c) This is just the logarthmic derivative of the above formula, which is how we define the logarthmic derivative anyway!

□

Y.Z.'s notes. I hate analysis. It's so annoying. □

Problem 2. *Sum the following two series $\sum f(n)$ by considering the contour integral $\int \pi f(z)\cot \pi z dz$.*

$$\sum_{n=1}^{\infty}\frac{1}{n^4+a^4}, \qquad \sum_{n=1}^{\infty}\frac{n^2}{n^4+a^4},$$

where a is a positive number.

Solution. These are basically the same problem, and you solve it with the same technique as given in the notes. If the poles are simple nonintegers (which they are not) at $a_1, \ldots, a_k$ and they have residues $b_1, \ldots, b_k$, and the degree of the denominator is at least 2 more than the numerator (which is true here!), then we can use the box integration trick with

$$\int \pi cot(\pi z) f(z) dz$$

which conveniently has a pole at exactly the integers to get

$$\sum_{-\infty}^{\infty} f(n) = -\pi \sum_{1}^{k} b_i cot(\pi a_i).$$

(a) We have exactly the poles $\pm\sqrt{\pm i}a$, and the corresponding residues $\mp\sqrt{\pm i}/4a^3$.

$$(1/4a^3)(e^{\pi i/4}\pi cot(e^{\pi i/4}a\pi) + coth(e^{\pi i/4}a\pi)) - 1/(2a^4)$$

(b) The poles are the same, but the residues are now $\mp\sqrt{\mp i}/4a$.

$$(\pi/4a)(e^{\pi 3i/4}cot(e^{\pi i/4}a\pi) + e^{\pi i/4}cot(e^{\pi 3i/4}a\pi))$$

□

Y.Z.'s notes. This is straight out of page 5 on the notes. Theoretically, you need to have checked the conditions, but I was very lenient. This is a good trick to learn regardless. I skipped the algebra since it was routine, and it really doesn't deserve any points taken off if you miss any anyway. □

Problem 3. *Let D be the domain in $\mathbb{C}$ which is the open strip $\{\, 0 < y < \pi \,\}$ minus the ray $\{\, x \leq 0,\ y = \frac{\pi}{2} \,\}$. Find a harmonic function u on D such that the boundary value of u on $\{\, x \leq 0,\ y = \frac{\pi}{2} \,\}$ is -1 and the boundary value of u on both $\{\, y = 0 \,\}$ and $\{\, y = \pi \,\}$ is 1. Hint: consider first the exponential map $z \mapsto e^z$.*

Solution. This is damn hard to describe, and I'm not nice enough to draw LaTeX pictures for you guys, so you have to accept a proof with words.

First, take $z \to e^z = z_1$. This should send you to the upper half plane minus the segment $[0, i]$, which has potential -1, and the real line, which has potential 1. Then take $z_1 \to z_1^2 = z_2$. This gets you the whole complex plane, minus $[0, -1]$ (potential -1) and $\mathbf{R}^+$ (potential 1).

Now, shift $z_2 + 1 = z_3$. You get the plane minus $[0, 1]$ (value -1) and $[1, \infty)$ (value 1). Take a square root with the branch giving angles $0, 2pi$ for $z_4 = \sqrt{z_3}$ that sends this to the upper half plane minus the real line, with $[-1, 1]$ having potential -1 and 1 otherwise.

These are the easy ones. The next one is slightly trickier, but you have done this before. Use $z_5 = (z_4 + 1)/(z_4 - 1)$ to send this to the lower half plane (lower since determinant is negative), with -1 on the negative real line and 1 on the positive real line. Finally, take a log to send the whole thing to the parallel strip between two adjacent multiples of π.

Now, many of you had sign differences, but I think this, the next step, is very important. Here's what I think will resolve the sign difference.

Note that whatever branch of log you pick, to be continuous, since we are mapping the lower half plane, the value of the argument needs to increase continuously from $(-\infty, 0)$ to $(0, \infty)$, meaning it has to increase π radians. Thus, the strip you choose needs to send the part with potential 1 (the positive real axis) higher than the strip with potential -1. You have no choice here! Therefore, the one that makes the most sense is to use the branch of log between $-\pi$ and 0 (you may also use a different branch which would then get a different answer). So now taking the log you have value 0 on the part you wish to be 1 and $-\pi$ on the part you wish to be -1. Thus, we may do the transformation of multiplying by $2/\pi$ and adding 1 to get:

$$u = (2/\pi) Im(log(\frac{\sqrt{e^{2z}+1}+1}{\sqrt{e^{2z}+1}-1})) + 1.$$

□

Y.Z.'s notes. Usually a sign error is just a sign error. But here I think it is vital to get the right sign if you really understand how your branch cut works. You'll get a different answer if you had picked a different log, of course. The reason why I stress it so much here is that in the final part (taking log) you simply cannot get a branch where the negative real axis goes to the top boundary while the postive real axis goes to the bottom boundary, as some of you did. Convince yourself why this is true. Another way to see this: if you have continuity but the negative real axis goes to the top boundary but the postitive real axis goes to the bottom boundary (i.e. you give the two the angles π and 0 respectively, or equivalent), then where does the negative imaginary axis go? It must get $3\pi/2$ (or bigger) or $-\pi/2$ (or smaller), but by continuity it must go between the two boundaries! □

Problem 4. *Let C_1 be the circle $|z| = 5$ and C_2 be the circle $|z-2| = 2$. Let D be the domain inside the circle C_1 and outside the circle C_2. Find a harmonic function u on D whose boundary value at C_1 is 1 and whose boundary value at C_2 is 0. Hint: consider the linear fractional transformation with real coefficients which maps the four points $-5, 0, 4, 5$ to the four points $-R, -1, 1, R$ with R to be determined.*

Solution. We take the hint. We want a linear fractional transformation that sends the real line to the real line and gets two concentric circles centered at the origin. By symmetry, we know that we can assume in such a transformation $-5, 0, 4, 5$ get sent to some $-R, -1, 1, R$. Since the cross-ratio is satisfied, we have

$$\begin{aligned}
\frac{(-5-4)(0-5)}{(-5-5)(0-4)} &= \frac{(-R-1)(-1-R)}{(-R-R)(-1-1)} && (1)\\
\frac{45}{40} &= \frac{R^2+2R+1}{4R} && (2)\\
9R/2 &= R^2+2R+1 && (3)\\
0 &= 2R^2-5R+2. && (4)
\end{aligned}$$

Solving, we can get $R = 2$ or $R = 1/2$. Let's pick $R = 2$ (this way we know the orientation is fixed with 0 potential inside and 1 outside, since $2 > 1$). Note that knowing R we can just read off the linear fractional transformation with professor Siu's technique in the notes by writing

$$\begin{aligned}
\frac{(-5-z)(0-5)}{(-5-5)(0-z)} &= \frac{(-2-w)(-1-2)}{(-2-2)(-1-w)} && (5)\\
\frac{25+5z}{10z} &= \frac{6+4w}{4+4w} && (6)\\
(2+2w)(5+z) &= (6+3w)(z) && (7)\\
10+10w+2zw+2z &= 6z+3zw && (8)\\
w &= (4z-10)/(10-z). && (9)
\end{aligned}$$

The rest is easy. You can either use use physics straight away (knowing that potential between two concentric circles is proportional to $-ln(r)$, where r is the radius), or you can do one more map (this is equivalent) and use ln, which sends the annulus to two lines parallel to the y-axis, in which case we know that the potential is just linear between the two lines. Using either method (again, they are equivalent, but you should convince yourself of this!), you solve and you get

$$u = 1/(log(2))Re[log(4z-10)/(10-z)].$$

Of course, the way you solve for the $1/(log(2))$ is to just make sure that your function $aRe(blah) + b$ has value 1 and 0 at two of the prescribed points. Here, testing at 0 and 5 suffices, since you should get 0 and 1 respectively.

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Y.Z.'s notes. Many of you used mathematica for some reason to find the explicit transformation? I think it is much better for abstract reasoning to come in and say: "well, we only have three degrees of freedom, and I know that $-5, 0, 5$ goes to $-2, 1, 2$. Thus, I also know $-5, 0, z, w$ goes to $-2, 1, w, 2$, and I can just use this to solve for the function."

Also, the reason I stress twice for you to convince that the "physics" way and the "math" way are equivalent is to show you that the way you learned in physics of integrating a radial potential function to get the potential is really just knowing that if you have parallel plates at 0 with potential 0 and 1 at potential 1, the potential at a between them is just a. I personally think it is very nice that these two facts, where the first one seems a lot more complicated than the second, is really just the same thing. ☐

Problem 5. *(from Ahlfors p.83, #6). Suppose that a linear fractional transformation carries one pair of concentric circles into another pair of concentric circles. Prove that the ratios of the radii must be the same.*

Solution. Probably the simplest way to approach this problem is seeing that the cross ratio is preserved and using the strucutre of a linear fractional transformation! Note that a line through two concentric circles must end up as being a line or a circle. But we know that the boundaries of the two annuli go to each other, and this line intersects each circle at two distinct points, so the only way that it can end up being a line or circle and still intersect each boundary circle at two distinct points is for it to be a line.

Now, you must convince yourself that the line goes through the annuli through the center. Many of you assumed this without proof, which is a bad habit. In fact, this is almost the whole problem. One way you can check this is to take two perpindicular lines through the center of the first annulus, show that it must go through two perpindicular lines, and through the cross ratio this cannot be preserved unless they intersect in the middle, meaning the center must go to the center.

Afterwards (now knowing that the center actually goes to the center), take one of the lines that go through the center of the first annulus. It must intersect the boundary of the annulus in 4 consecutive points. Call them a, b, c, d. Then we know that $\frac{(a-c)(b-d)}{(a-d)(b-c)}$ will be preserved. But this is just $(R+r)^2/(Rr) = 2 + R/r + r/R$, where R is the larger radius and r is the inner radius. For this to be preserved is the same thing as saying "the ratio of the larger and smaller radii is preserved," which is what we want.

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Y.Z.'s notes. Many of you also approached this problem by thinking of the map as a compostion of a translation, an inversion, and a rotation. This is also fine, but takes significantly longer to write up. Most of you did not prove that the center goes to the center, which is a very key part of the problem.

Another cute solution by one of you works like this: take a tangent to the inner circle. Draw two lines from the center: one goes to the tangent point, and one goes to where the tangent meets the outer circle. These two lines make some angle θ, and you know directly from the picture that the ratio is just $cos(\theta)$. Also, since we know the center goes to the center, after the map these two lines just either stay as drawn or swap. But since the map is conformal θ is unchanged - meaning the ratio stays the same. Of course, this does not work if you don't show that the center goes to the center! ☐