Math 115
Fall 2006 Assignment 3 Solution Yan Zhang

Problem 1. *(partly from Ahlfors p.161, #3). Evaluate the following integrals by the method of residues:*

(a)
$$\int_{-\infty}^{\infty} \frac{x^2 - x + 2}{x^4 + 10x^2 + 9}\, dx$$

(b)
$$\int_0^{\infty} \frac{x \sin x\, dx}{(x^2 + a^2)^2} \quad (a \in \mathbb{R})$$

(c)
$$\int_0^{\infty} \frac{\sin^2 x}{x^2}\, dx$$

(Hint: Convert $\sin^2 x$ *to* $\cos 2x$ *first.)*

Solution. (a) We use the classical half-circle contour, from $-R$ to R on the real axis then a big semicircle from R back to $-R$. Note that since the order of the function is $1/x^2$ as $|R| \Rightarrow \infty$, we may ignore the semicircle. Thus, we just need to calculate the residues in the upper-half plane.

Factor the expression (now in z for example, as a function of a complex variable) as $\frac{(x^2-x+2)}{(x+3i)(x-3i)(x+i)(x-i)}$. Note that the two things that contribute are $3i$ and i. Now consider the residues at i and $3i$. Since they are order 1, we may just ignore the corresponding term in the denominator and plug in the value, to get $\frac{1-i}{(i+3i)(i-3i)(i+i)} = \frac{1-i}{16i}$ for i and $\frac{-7-3i}{(3i+3i)(3i+i)(3i-i)} = \frac{7+3i}{48i}$ for $3i$. Adding and multiplying by $2\pi i$, we get $5/12i$.

(b) We again use the half-circle contour. Notice that since our function is even, our desired answer (call it I) is just half of the function integrated through the whole real axis. Now we do the same thing as above, except we are really integrating $\frac{ze^{iz}}{(z^2+a^2)^2}$ and then taking the imaginary part. We have two order two poles, but only the pole $|a|i$ matters. Let $b = |a|$ We we want

$$(1/2)Im(2piid/dz(ze^{iz}/(z-bi)^2)_{z=ib}) = \pi e^{-b}/4b$$

(c) We take the hint and write this as $\int_0^{\infty}(1 - cos(2x))/x^2 dx$. By integration by parts, using $1/x^2 dx = dv$, $v = -1/x$, $u = 1 - cos(2x)$, and $du = 2sin(2x)$, we get $[(1 - cos(2x))(-1/x)]_0^{\infty} + \int(sin(2x))/xdx$. The first part goes to 0 (read notes below to see why! This is important). The second part we think of as $Im(\int e^{2ix}/xdx)$, and use similar methods as in class (we do a half circle but jump over 0, and that's the only place where we gain anything) to get $\pi/2$.

□

Y.Z.'s notes. Basically everyone got this one. I still felt like doing it since this is really fundamental (and probably will be on the midterm. Hint hint?). On (b) many of you guys forgot to consider the case for a negative, though I think that is a really minor point. On (c) however, way too many people argued that $((1 - cos(2x))/x)$ approached 0 at both ∞ and 0. This is very dubious - the first limit is clear, but in the second limit both top and bottom go to 0, so how do you just know that? There are two quick ways to fix this: one is to use L'Hopital's rule (you remember?) and get $2sin(2x)/1$, which now definitely goes to 0 so you are fine. The other is to write it as a power series expansion in x and note that the numerator wins. This is equivalent to the first method (why?). Make sure you get used to making arguments like this. □

Problem 2. *(partly from Ahlfors p.161, #3). Evaluate the following integrals by applying the method of residues to branches of holomorphic functions.*

(*a*)
$$\int_0^\infty \frac{x^{\frac{1}{3}}}{1+x^2}\,dx$$

(*b*)
$$\int_0^\infty \frac{\log\left(1+x^2\right)\,dx}{x^{1+\alpha}} \quad (0<\alpha<2)$$

(Hint: Try integration by parts.)

(*c*)
$$\int_0^\infty \frac{\left(\log x\right)^2 dx}{1+x^2}$$

Solution. (a) We do a half-circle. Note that you need to pick a branch - the usual between 0 and π is fine. Nothing is really hard about this - we get $\pi/\sqrt{3}$ by calculating the only residue we have is at i.

(b) Many of you did the whole circle here, though half-circle is probably easier since you don't have to calculate both residues. First, let $u = ln(1+x^2)$ and $dv = x^{-a-1}dx$ and do integration by parts. We should get $[ln(1+x^2)(-1/a)x^{-a}]_0^\infty + (2/a)\int_0^\infty \int \frac{1}{(1+x^2)(x^{a-1})}dx$. The first part is 0 for similar reasons as we treated problem 1, part c.

Now, do the coveted half-circle (jumping over 0, of course - we have a pole there). Both semi-circles go to 0, happily. Our single residue at i gives $(2\pi i)i^{1-a}/2i = \pi i^{1-a}$.

Let's call the right line integral I (which goes to what we want). The left one is inverted, so do a $x \Rightarrow -x = e^{\pi i}x$:

$$\int_{-R}^{-r} x^{1-a}/(1+x^2)dx = \int_r^R (-x)^{1-a}/(1+x^2)dx,$$

We should pull out $(-1)^{1-a}$ coming from the negative sign. Thus, we know that $I(1+(-1)^{1-a}) = \pi i^{1-a}$. So

$$\begin{aligned} I &= \pi e^{i\pi/2(1-a)}/(1+e^{i\pi(1-a)}) && (1)\\ &= \pi/(e^{-i\pi/2(1-a)}+e^{i\pi/2(1-a)}) && (2)\\ &= \pi/(2cos(\pi/2-\pi a/2)) && (3)\\ &= \pi/(2sin(a\pi/2)). && (4)\end{aligned}$$

(c) Use the same contour. Again, the small and big semicircles go to 0 (since polynomial beats log when $|z|$ is big, and when $|z|$ is small, we know $(log(z))^2|z|$ goes to 0 so we are fine). The residue is $(2\pi i)(log(i))^2/2i = -\pi^3/4$. Now

$$\int_{-R}^{-r} (log(z))^2/(1+z^2)dz = \int_r^R ((log(z)+\pi)^2)/(1+z^2)dz,$$

Which has real part exactly equal to $I + \int_r^R \pi^2/(1+z^2)dz$. So we know that $2I + \pi^2 arctan(z)|_0^\infty - \pi^3/4$, and we solve to get $I = \pi^3/8$.

□

Y.Z.'s notes. Think carefully about why $log(x)/(1+x^2)$ dies when you integrate it on the small circle! The integrand actually does NOT go to 0, it goes to infinity! The reason the integral goes to 0 is that we need to multiply by an extra term $|x|$ to estimate the integral, which is a polynomial. And polynomials destroy logarithms utterly. □

Problem 3. *Use the branch cut* $[0,1]$ *and the theory of residues to establish the following formula of integration.*

$$\int_0^1 \frac{\sqrt[4]{x(1-x)^3}}{(1+x)^3}\,dx = \frac{3\pi\sqrt[4]{2}}{64}.$$

Hint: Follow the techniques used for the in-class computation of

$$\int_0^1 \frac{dx}{x^\alpha(1-x)^{1-\alpha}} = \frac{\pi}{\sin\alpha\pi} \quad (0<\alpha<1).$$

Solution. Here's an example of a place where you get two branchs that "meet up" nicely. Cut thte branch for $z = 1/4$ at $[0,\infty)$ with angle 0 to 2π and cut the branch for $(1-z)^{3/4}$ at $[-\infty, 0]$ with angle $-\pi$ to 2π (which becomes a cut at $[1,\infty)$). Note that these definitions meet up nicely at $(1,\infty)$, so our only problem is the segment $[0,1]$.

Think about what this means. This means that, using our "usual" idea of the plane as being from 0 to 2π: for the z, we are using the "usual" argument. But for the $(1-z)$, we take the arguments (centered at 1) such that it is $-\pi$ (NOT π) a bit above, say, 2, and loops positively around 1 to π (NOT $-\pi$) when it is a bit below 2. Note this is the case since we did the map $(1-z)$ which does a shift and then rotates your plane 180 degrees.

Anyways, take a loop around the segment as shown in class. We should get the negative of the residue at -1 (the reverse of our loop "surrounds" the outside of our bad segment, which has the only pole there), which ends up being $2\pi i(1/2)(d^2/dz^2(z^{1/4}(1-z)^{3/4}))|_{-1}$. This equals $-\pi i(3/32)(-1/2)^{1/4}$

The little half-circles around 0 and 1 can be safely ignored. But we cannot ignore the line segments! Around the bottom we have $(2\pi, 0)$ respectively for the two angles corresponding to z and $(1-z)$, and on top we have $(0,0)$ respectively. You can basically now follow the Professor's notes: you should be dividing the above value from the residue by $(1-e^{-\pi/2})$ (which is the factor you get after you work out the angles) to get $3\pi 2^{1/4}/64$.

□

Y.Z.'s notes. We go around the loop counterclockwise, we should get the negative sum of the residues outside. Similarly, we could have just gone around the loop clockwise. Why?

Also, many of you copied what the professor did in the notes and thought that around the bottom the angles are $(2\pi, \pi)$ and around the top the arounds are $(0, -\pi)$ or something like that - this is incorrect. Think about it - you want your value to be real on the x-axis, and according to your current scheme it absolutely cannot be real there. □

Problem 4. *Let D be the domain in $\mathbb{C}$ formed from $\mathbb{C}$ by deleting the three line-segments $[-1, i]$, $[1, i]$ and $\{x = 0, y \geq 1\}$. For $a \in D$ let $f(a)$ be defined by*

$$f(a) = \int_{C_a} \frac{-2z}{1-z^2}\, dz,$$

where C_a is any piecewise smooth curve in D joining 0 to a.

(a) Show by Cauchy's theorem for holomorphic functions that the definition of $f(a)$ is independent of the choice of the piecewise smooth curve C_a in D.

(b) Show that $f(z)$ is a branch of $\log(1-z^2)$ on D in the sense that $f(z)$ is holomorphic on D and $e^{f(z)} = 1 - z^2$ for $z \in D$.

(c) Compute $f(2)$.

Solution. (a) This works for any simply-connected space, and our domain is simply-connected (that is to say, any loop can be contracted to a point). The reason this is true is that suppose we have two paths from 0 to z. Then by Cauchy's Theorem the integral over the first path plus the integral over the reverse of the second path is a closed loop which is holomorphic inside, so we must get 0. Hence, the two integrals are the same.

(b) Integrals of holomorphic functions are holomorphic. This is a well-known and well-used fact, similar to the fact that derivatives of holomorphic functions are holomorphic. The way to think about this is that "holomorphic" basically means "has a local power series" (meaning at each point there is a neighborhood where you can write a power series), so if you have a local power series, you can obviously integrate/differentiate it to get a local power series.

The second part makes you show that $e^{f(z)} = 1 - z^2$. Here's a cute solution that I showed during section, but I'll reproduce: Write $g(z) = 1 - z^2$. Showing the above is true is equivalent to showing that $H(z) = e^{f(z)}/g(z) = 1$. Note that this is equivalent to showing that $H'(z) = 0$ and $H(z) = 1$ at any point. The former you can do since you know how to take a derivative of a function defined via an integral by the fundamental theorem of calculus. The latter is easy - throw in $z = 0$.

(c) This part gave everyone problems. Think about it this way. Take the contour that makes a small semicircle from 0 to 2, dodging the pole at 1. Now, $ln(1 - z^2) = ln(1 - z) + ln(1 + z)$. Of course, we must define what the later two ln's are. Note that the one restraint we have is that their arguments at 0 must add to 0, since we DEFINED f to be 0 at 0.

So what happens when we move from 0 to 2? We look at what happens corresponding to the two poles at -1 and 1. For the pole at 1, the radius remains constant, so we don't gain anything in the real part. However, we did a π rotation, so we must pick up πi for the imaginary part. Similarly, for the pole at -1, the radius changes from 1 to 3, so we gained $ln(3) - ln(1) = ln(3)$ in the real part. However, the angles of 2 and 0 corresponding to -1 are the same, so we don't gain any imaginary part. Thus, the answer is $ln(3) + \pi i$.

□

Y.Z.'s notes. For (a), I don't think you have the topological tools to argue that this space is simply-connected, but for those of you interested in learning a little topology, pick up Munkres or something and do some reading. For (b) most of the answers are bogus, and involves using $ln(1 - z^2)$ haphazardly without knowing what it means. The problem is basically asking you that you can write our function as $ln(1 - z^2)$, so you really cannot use $ln(1 - z^2)$ anywhere in the proof reasonably without being guilty of circular reasoning. Convince yourself of this. For (c) I did not receive any satisfactory complete solution to this problem. Please make sure you understand branch cuts! Ask me or the prof to set up a meeting if you have to. This may be the most useful yet most confusing abstract issue in the complex analysis you will learn from this course. □