Math 115
Fall 2006 Assignment 2 Solution Yan Zhang

Problem 1. *Problem 1 (from Ahlfors p.108, #2, #7; p.120, #1, #3). (a) Let $r > 0$ and x be the real part of the complex variable z. Compute*

$$\oint_{|z|=r} x\,dz$$

for the positive sense of the circle in two ways: first by use of a parameter, and second, by observing that

$$x = \frac{1}{2}(z + \bar{z}) = \frac{1}{2}\left(z + \frac{r^2}{z}\right)$$

on the circle.

(b) Suppose $P(z)$ is a polynomial of a single complex variable with complex coefficients and C is the circle $|z-a| = R$, where $a \in \mathbb{C}$ and $R > 0$. Show that

$$\int_C P(z) d\bar{z} = -2\pi i R^2 P'(a).$$

(c) Compute

$$\oint_{|z|=1} \frac{e^z}{z} dz$$

(d) Let $\rho > 0$ and $a \in \mathbb{C}$ with $|a| \neq \rho$. Compute

$$\oint_{|z|=\rho} \frac{|dz|}{|z-a|^2}.$$

Hint: make use of the equation $z\bar{z} = \rho^2$ and

$$|dz| = -i\rho \frac{dz}{z}.$$

(e) Verify that

$$\oint_{|z|=4} \frac{z^{15}}{(z^2+1)^2 (z^4+2)^3} dz = 2\pi i (error?)$$

by using the change of variables $z = \frac{1}{w}$.

Solution. (a) First way: write $x = rcos(\theta)$. Then $dz = dx + idy = [ircos(\theta) - rsin(\theta)d\theta]$. Hence, you are integrating

$$\int_0^{2\pi} r^2 cos(\theta)(icos(\theta) - sin(\theta))d\theta = \int_0^{2\pi} r^2 (icos^2(\theta) - sin(2\theta)/2)d\theta \quad (1)$$

$$= \int_0^{2\pi} r^2 (i(1 + 2cos(2\theta))/2 - sin(2\theta)/2)d\theta \quad (2)$$

$$(3)$$

We obviously get $r^2\pi i$ from the constant term. Since the other two terms integrate to trigonomic functions, they vanish on our curve. So the answer is $r^2\pi i$. The second way is much easier: just integrate $(1/2)\int(z + r^2/z)dz$ as they say. The first part is holomorphic, so we can ignore it. The second is just $r^2(2\pi i)$ since we know how to integrate $1/z$. Thus, we get $r^2\pi i$.

(b) On this circle, $(\overline{z} - a) = R^2/(z - a)$. Thus, $d\overline{z} = -R^2(z - a)^2$. So we can just integrate $-R^2\int P(z)/(z - a)^2dz$. Now (this is important), start with the Cauchy's integral formula we know and love,

$$P(a) = 1/(2\pi i)\int P(z)/(z - a)dz,$$

now differntiate under the integral with respect to a to get $P'(a) = 1/(2\pi i)\int P(z)/(z-a)^2dz$. The right side is just $(\int P(z)d\overline{z})/(-R^2)$, so we are done.

(c) Just use Cauchy's integral formula to get $2\pi i$. This amounts to evaluating e^z at 0 to get 1, then multiplying by the ubiquitous $2\pi i$ factor.

(d) The hard part is to understand what $|dz|$ means. As in the hint, $|dz| = -i\rho dz/z$, where ρ is the magnitude of z (there are many ways to check this - the easiest I can think of is to to just see that

$$\begin{aligned}
-i\rho dz/z &= -i\rho(i\rho e^{i\theta}d\theta)/(\rho e^{i\theta}) &(4)\\
&= \rho d\theta &(5)\\
&= \rho|d\theta| &(6)\\
&= |d(re^{i\theta})| = |dz|, &(7)
\end{aligned}$$

After getting this the rest is routine. You have

$$\begin{aligned}
\int \frac{-i\rho}{z(z-a)(\overline{z}-\overline{a})}dz &= \frac{-i\rho}{z(z-a)(\rho^2/z-\overline{a})}dz &(8)\\
&= \frac{-i\rho}{(z-a)(\rho^2-|a|^2z/a)}dz &(9)\\
&= \frac{i\rho a/|a|^2}{(z-a)(z-a\rho^2/|a|^2)}dz &(10)\\
& &(11)
\end{aligned}$$

Then you note only one is a pole that you care about, depending on $|a|$. You should get two answers, which you can both write as $(2\pi\rho)/(||a|^2 - \rho^2)|$.

(e) I think all of you got this. Except you need to be careful - when you do the change of variables, the direction of the contour changes! So you need an extra negative sign. The right answer should be $-2\pi i$. The professor was off by a negative sign, I think. □

Y.Z.'s notes. The technique I use in the second part is called "parametric differentiation." This is allowed if the derivative of your function is continuous and absolutely convergent - and in this case it is pretty clear since the only problem is at a and we don't get close to it at all. Of course, I didn't need to do it in this case - I just derived Cauchy's integral formula extended to derivatives, which you may or may not have done in class (But you should know it!). The technique of parametric differentiation, however, is a favorite of the late Ricahrd Feynman and participants in the Putnam competition, so it might be good to know. To see why this is legal, read, say, Rudin's Principles of Mathematical Analysis. Also, part (e) is not there for routine calculation - this is an important trick. Note we are switching the poles on the inside of the contour with the poles outside of the contour (in this case, at infinity), since the ones on the inside really suck. □

Problem 2. *(from Ahlfors p.108, #5). Suppose γ is a smooth closed curve in $\mathbb{C}$ and D is a domain in $\mathbb{C}$ which contains γ. Suppose $f(z)$ is a holomorphic function on D. Show that*

$$\int_\gamma \overline{f(z)} f'(z) dz$$

is purely imaginary.

Solution. As in the hint, know that the exact differential always integrates to 0. So we know the integral of $\overline{f'(z)} f(z) + f'(z)\overline{f(z)} dz$ is zero (we implicitly use the fact that differentiation commutes with conjugation, which is sortakindanotreally obvious. But note now that integration commutes with conjugation, so if the integral we desire is I, we know that $I + \overline{I} = 0$. Thus, I is completely imaginary. □

Y.Z.'s notes. Conjugation basically commutes with everything. This has a deeper meaning in mathematics on why this operation is so special that other operations cannot "detect" it. I don't know all the details myself.

Anyway - what do we learn here? Learn that exact differentials always integrate to 0 and abuse this fact. Also, if you want to prove something is purely imaginary, see if you can show the sum of it and its conjugate is 0. □

Problem 3. *(from Ahlfors p.130, #5). Prove that an isolated singularity of $f(z)$ is removable as soon as either $\operatorname{Re} f(z)$ or $\operatorname{Im} f(z)$ is bounded above or below. Hint: Apply fractional linear transformation*

$$w \mapsto \frac{aw + b}{cw + d}$$

(with $a, b, c, d \in \mathbb{C}$ and $ad - bc \neq 0$).

Solution. Use the fact that an isolated singularity is removable if the function is bounded on some punctured disk centered at the bad point. Now the hint should make sense. Without loss of generality, say that the real part is bounded above (the other 3 cases are all really the same).

In this case, note we can assume it is bounded above by 0 (since we can move the function by a constant). Then $(w-1)/(w+1)$ sends the function to a holomorphic function with absolute value no more than 1, and we are immediately done by the assertion given above. □

Y.Z.'s notes. Note this foreshadows conformal mapping. Conformal mapping in general is a tool that lets you think about a function on a region in a different light by doing a reversible transformation onto another region, so you are not losing any information. This concept of shifting the frame to make the problem easier is a very important general technique. For just problem set 2, however, I thought this problem was pretty hard. □

Problem 4. *Let $n \in \mathbb{N}$. Suppose $f(z)$ is a holomorphic function on all of $\mathbb{C}$. Suppose for every $\varepsilon > 0$ there exists $A_\varepsilon > 0$ such that*

$$|f(z)| \leq \varepsilon\, |z|^{n+1} + A_\varepsilon \text{ for } z \in \mathbb{C}$$

is satisfied. Show that $f(z)$ is a polynomial of degree at most n.

Hint: Use

$$f^{(n+1)}(z) = \frac{(n+1)!}{2\pi i} \int_{|\zeta|=R} \frac{f(\zeta)d\zeta}{(\zeta - z)^{n+2}} \quad \text{for } |z| < R$$

and

$$\left| \int_C g(z)dz \right| \leq \left(\sup_{z \in C} |g| \right) \cdot (\text{length of } C)$$

to show by letting $R \to \infty$ and then $\varepsilon \to 0$ that $f^{(n+1)}(z) = 0$ for every $z \in \mathbb{C}$.

Solution. The hint basically does all the problem for you. Note that $f^{(n+1)}(z) = 0$ for all z implies that f is a polynomial of degree $\leq n$, since you can just integrate $n+1$ times and see what f must look like.

Take the formula given (which, as you remember, is the extended Cauchy's integral formula). Note that as $R \Rightarrow \infty$, the denominator of the integrand has same order as z^{n+2}. Thus we may use the inequality to get

$$|f^{(n+1)}(z)| \leq lim_{R \to \infty} |(n+1)!/(2\pi)| (\epsilon |R|^{n+1} + A_\epsilon)(2\pi R)/R^{n+1}$$

Here, we can switch integration and limit since we have absolute convergence. This then is bounded by some constant multiple of ϵ. Now, we can let ϵ go to 0 to show that this derivative must actually be identically 0, so we are done by the previous remark.

□

Y.Z.'s notes. Again, I reiterate that you can switch integration and limit when you have absolute convergence. This is the same as when you can switch around double summations, and even integrals and integrals (Recall my blurb on parametric differentiation in problem 1). One of the underlying reasons of all of this is Fubini's Theorem, which allows you to switch integrals when they "make sense."

This problem is actually sort of important: it demonstrates three very important principles: switching of integration and limit, bounding the integral by bounding its integrand and then multiplying by the length, and the extended Cauchy's formula. These are very simple but good to know well. □

Problem 5. *The coefficient of the n-th power of z in the power series expansion, about $z = 0$, of the function*

$$f(z) = \frac{4 - z^2}{4 - 4zt + z^2} \quad (-1 \leq t \leq 1)$$

is called a Tchebychev polynomial (notation: $T_n(t)$). Prove that

$$T_n(t) = \frac{1}{2^{n-1}} \cos\left(n \cos^{-1} t\right).$$

Solution. This solution is sick:

The first part is rewriting the question in a way that is easier to answer. Notice that proving that those terms are the Chebeshev polynomials is equivalent to saying that (let $t = cosx$, which we can do since we know $-1 < t < 1$):

$$\begin{aligned}
\sum_{n=0}^{\infty} 1/2^{n-1}cos(nx)z^n &= \sum_{n=0}^{\infty} 2(Re1/2^n e^{inx}z^n) &(12)\\
&= 2Re(\sum_{n=0}^{\infty} 1/2^n(ze^{ix})^n) &(13)\\
&= Re(2\sum_{n=0}^{\infty}(ze^{ix}/2)^n) - 1 &(14)\\
&= Re(2/(1 - ze^{ix}/2)) - 1 &(15)\\
&= Re(\frac{2 - ze^{-ix}}{(1 - ze^{ix}/2)(1 - ze^{-ix}/2} &(16)\\
&= \frac{2 - z\cos(x)}{1 - z\cos(x) + z^2/4} &(17)\\
&= \frac{2 - zt}{1 - zt + z^2/4} &(18)\\
&= \frac{8 - 4zt}{4 - 4zt + z^2} &(19)\\
& &(20)
\end{aligned}$$

The taking real part at beginning looks dubious, but don't let the letter z scare you! Note that we are writing this is the power series of a real variable z. Now note that the function I have at the end is actually what we want - it is $1 + f(z)$, the given function in the hw, but clearly the given function is 1 off, since it has constant term of 1, but the term $T_0(t) = 2$. The function we have here has constant term $8/4 = 2$, so this one is correct.

□

Y.Z.'s notes. This solution is actually very similar to problem 7 of the previous problem set. The things to learn here: think about what the problem MEANS - it is equivalent to saying if you add up the power series as above, you'd get the function given. □