Problem 1. *(#18 on Page 52 of Gelfand and Fomin). Find the extremals of the functional*

$$J[y] = \int_0^1 \left(y'^2 + x^2\right) dx,$$

subject to the conditions

$$y(0) = 0, \quad y(1) = 0, \quad \int_0^1 y^2 dx = 2.$$

Solution. We use the functional/integral analogue of Lagrange multipliers:

$$\begin{aligned} F_y - \frac{\partial}{\partial x} F_{y'} &= \lambda(G_y - \frac{\partial}{\partial x} G_{y'}) \\ -\frac{\partial}{\partial x}(2y') &= \lambda 2y \\ y'' + \lambda y &= 0. \end{aligned}$$

Solving, $y = C_1 e^{i\sqrt{\lambda}x} + C_2 e^{-i\sqrt{\lambda}x}$. Using $y(0) = y(10) = 0$, we get $y = c_1 \sin(n\pi x)$. But we also have the constraint that $\int c_1^2 \sin^2 n\pi x dx = 2$, so $c_1 = \pm 2$ and we have $y = \pm 2 \sin(n\pi x)$. □

Problem 2. *(#1 on Page 94 of Gelfand and Fomin). Use the canonical differential equations to find the extremals of the functional*

$$\int \sqrt{x^2 + y^2} \sqrt{1 + y'^2}\, dx,$$

and verify that they are of the form

$$x^2 \cos\alpha + 2xy \sin\alpha - y^2 \cos\alpha = \beta,$$

where α and β are constants.

Hint: The Hamiltonian is

$$H(x, y, p) = -\sqrt{x^2 + y^2 - p^2},$$

and the corresponding canonical system

$$\frac{dp}{dx} = \frac{y}{\sqrt{x^2 + y^2 - p^2}}, \quad \frac{dy}{dx} = \frac{p}{\sqrt{x^2 + y^2 - p^2}}$$

has the first integral

$$p^2 - y^2 = C^2,$$

where C is a constant.

Solution. To get to the hint, just compute $p = \frac{\partial F}{\partial y'} = \frac{y'\sqrt{x^2+y^2}}{1+(y')^2}$. Note this also gives an expresion of y'. Then $H(x, y, p) = y'p - f(x, y, y') = -\sqrt{x^2 + y^2 - p^2}$. The first integral then directly gets $p^2 - y^2 = C^2$. This gives you the differential equation

$$(y')^2(x^2 + y^2) - y^2(1 + (y')^2) = C^2(1 + (y')^2).$$

Some of you got stuck here - the key is to separate this equation and let $y = c\sinh(\theta)$ and $x = c\cosh\phi$. Integrating both sides then gives

$$\sinh^{-1}(y/c) - \cosh^{-1}(x/c) = c_1$$

for some c_1. Taking sinh of both sides gives the relation

$$xy - \sqrt{(c^2 + y^2)(x^2 - c^2)} = c_2c^2,$$

for some c_2 after some algebra. More algebra gives

$$c = (x^2 - y^2)/\sqrt{c_2^2 + 1} + 2xyc_2/\sqrt{c_2^2 + 1}.$$

We can then set $1/\sqrt{c_2^2 + 1} = \cos(\alpha)$ and $c = \beta$ to get

$$\beta = x^2\cos(\alpha) + 2xy\sin(\alpha) - y^2\cos(\alpha).$$

□

Y.Z.'s notes. Many of you guys used the hint directly, though it would be nice to know how the steps are derived. □

Problem 3. *(#5 on Page 95 of Gelfand and Fomin). Consider the variation problem for functional*

$$J[r, \theta] = \int_{t_0}^{t_1} \left[\frac{m}{2}\left(\dot{r}^2 + r^2\dot{\theta}^2\right) + \frac{k}{r}\right] dt,$$

which comes from the problem of the plane motion of a particle of mass m attracted to the origin by a force $\frac{k}{r^2}$. For this variation problem the radius vector r and the angle θ are the dependent variables and the time t is the independent variable. The symbol $\dot{r}$ denotes the derivative of r with respect to t and the symbol $\dot{\theta}$ denotes the derivative of θ with respect to t.

Verify that the functional $J[r, \theta]$ is invariant under rotations (which are given by $\theta \mapsto \theta + \alpha$ with r unchanged and $\alpha \in \mathbb{R}$). Use Noether's theorem (in polar coordinates) to verify Kepler's law that the line segment joining the particle to the origin sweeps out equal areas in equal times.

Solution. Since θ doesn't appear and only θ' does, the translation keeps the functional constant. Doing this shift and using Noether gives

$$F_{\theta'}\delta\theta + (F - \theta' F_{\theta'})\delta t = c.$$

Our parameter of transformation, indexed by ϵ, sends (r, θ, t) to $(r, \theta + \epsilon, t)$. Thus, $\delta t = 0$ since t is constant, while $\delta\theta = 1$ since θ changes linearly with coefficient 1. Thus, we have $F_{\theta'}$ equalling a constant. But this equals $mr^2\theta'$. This corresponds to constant area being swept out in equal times.

□

Y.Z.'s notes. Note our machinery allows us to prove this classical fact very quickly. Of course, one could have done this with just classical mechanics, as I'm sure many of you have. □

Problem 4. *(#7 on Page 95 of Gelfand and Fomin). Write and solve the Hamilton-Jacobi partial differential equation corresponding to the functional*

$$J[y] = \int_{x_0}^{x_1} f(y)\sqrt{1 + y'^2}\, dx,$$

and use the result to find the extremals of $J[y]$.

Hint: The Hamilton-Jacobi partial differential equation is

$$\left(\frac{\partial S}{\partial x}\right)^2 + \left(\frac{\partial S}{\partial y}\right)^2 = f^2(y),$$

with solution

$$S = \alpha x + \int_{y_0}^{y} \sqrt{f^2(\eta) - \alpha^2}\, d\eta + \beta.$$

The extremals are

$$x - \alpha \int_{y_0}^{y} \frac{d\eta}{\sqrt{f^2(\eta) - \alpha^2}} = \text{constant}.$$

Solution. Again, we need to get to the hint first with some simple calculation.

$$p = F_{y'} = \frac{y' f(y)}{\sqrt{1 + (y')^2}}$$

is the first important calculation. Then $H = py' - F = \frac{-f(y)}{\sqrt{1+(y')^2}}$. An easy calculation gives

$$\left(\frac{\partial S}{\partial x}\right)^2 + \left(\frac{\partial S}{\partial y}\right)^2 = f^2(y),$$

which has the solution given in the hint. To see this, you can set $\frac{\partial S}{\partial x}$ to a constant α, so $S = \alpha x + s(y)$ for some function s which only depends on y.

To get the extremals, note that page 4 of the lecture notes on Canonical transformations mention that the extremals are given by $\frac{\partial S}{\partial \alpha}$ equalling a constant (as long as α is chosen to those suitable conditions, which are easy to check). This follows trivially from the last step. □

Y.Z.'s notes. The choosing parameter to be whatever you want type of thing is bogey-looking, but mathematically sound. It is something like gaining an extra degree of freedom. □

Problem 5. *(#11 on Page 130 of Gelfand and Fomin). Prove that the extremal*

$$y = \frac{y_1 x}{x_1}$$

corresponds to a local minimum of both functionals

$$\int_0^{x_1} \frac{dx}{y'}, \quad \int_0^{x_1} \frac{dx}{y'^2},$$

where $y(0) = 0$, $y(x_1) = y_1$, $x_1 > 0$, $y_1 > 0$.

Solution. The Euler-Lagrange equations $F_y - \frac{d}{dx} F_{y'} = 0$ gives that $y'' = 0$ for both cases, or $y = ax + b$. $y(0) = 0$ means that $y = ax$ for both, and $y(x_1) = (y_1)$ gives that $y = \frac{y_1 x}{x_1}$ as desired.

Now, we need to check that $P(x) = F_{y'y'}$ is positive. In both cases this is clearly true - we get $2/(y')^3$ and $6/(y')^4$ respectively. Also, note that $Q = F_{yy} - \frac{d}{dx} F_{yy'} = 0$ for both.

Finally, we just need to make sure that there are no points conjugate to 0. For a point x_2 to be conjugate we need some nonzero solution $h(x)$ on $[0, x_2]$ such that $0 = Qh = \frac{d}{dx}(Ph')$ and $h(0) = h(x_2) = 0$.

The first equality gives $2h''(y')^3 - 6(y')^2 h' h'' = 0$ for the first case, which gives $h'' = 0$, so $h = Cx + D$ for some C and D. The endpoints conditions give $h = 0$. A similar analysis for the second case gives the same result. □

Y.Z.'s notes. □

Problem 6. *(#15 on Page 130 of Gelfand and Fomin). Consider the catenary*

$$y = c \cosh\left(\frac{x + b}{c}\right)$$

(where b and c are constants), which is an extremal for the variational problem of the functional

$$J = \int_{x_0}^{x_1} y\sqrt{1 + y'^2}\, dx.$$

Show that any point on the catenary except the vertex $(-b, c)$ has one and only one conjugate point, and show that the tangents to any pair of conjugate points intersect on the x-axis.

Solution. Suppose we are looking for a family of catenaries through the point (p, r). Then we need $c\cosh(\frac{p+b}{c}) = r$, or $p = c\cosh^{-1}(\frac{r}{c}) - b$, or $b = c\cosh^{-1}(\frac{r}{c}) - p$. So we have a 1-parameter family of solutions to the E-L equations of the form

$$y = t\cosh(\frac{x + tcosh^{-1}(\frac{r}{t}) - p}{t})$$

note that when $t = c$ we get back the original catenary

$$y = c\cosh(\frac{x+b}{c}).$$

Now, to get h, we differentiate the general form respect to t, and let $t \to c$. Skipping some algebra, this gives us:

$$h = \coth(\frac{x+b}{c}) - (x-p)/c - \coth(\frac{p+b}{c})$$

(where we are going to need the fact that $r = c\cosh(\frac{p+b}{c})$). Note that when $x = p$ we get $h(p) = 0$, as we want. So for a point to be conjugate there needs to be another point with $h(p) = 0$. Solving, there is exactly one other point (unless $x = b$), the reason of which I leave as an exercise.

The cute thing, however, is that a point on the catenary has form $(x, c\cosh(\frac{x+b}{c}))$ and at that point has slope $\sinh(\frac{x+b}{c})$. Therefore, where the tangent intersects the x-axis is exactly when at $x - c\tanh(\frac{x+b}{c})$. However, note that the expression we just got for h means that when $h = 0$, we have exactly

$$x - c\coth(\frac{x+b}{c}) = p - c\coth(\frac{p+b}{c}),$$

which means the two tangents intersect on the x-axis.

□

Y.Z.'s notes. Many of you tried using equation (†) explicitly in the notes. This approach mirrors page 2 of the notes, which gives an explit construction of h. Congrats, you are done with the homework of Math 115. □