

Problem 1. (from Ahlfors p.2, #1 and p.4, #1). Compute the real part and the imaginary part of the following two complex numbers in terms of rational numbers and using the process of taking the square roots of positive numbers.

(a) $\left(\frac{2+i}{3-2i}\right)^2$.

(b) $\sqrt{\frac{1-i\sqrt{3}}{2}}$.

Solution. □

Y.Z.'s notes. Everyone got this one - luckily. □

Problem 2. (from Ahlfors p.6, #1). Show that the system of all matrices of the special form

$$\begin{pmatrix} \alpha & \beta \\ -\beta & \alpha \end{pmatrix},$$

combined by matrix addition and matrix multiplication, is isomorphic to the field of complex numbers.

Solution. Everyone got this one - basically. □

Y.Z.'s notes. I took off a couple of points when you only did things like closure under addition or multiplication. Somewhere in your proof you need to give me a map f that takes a complex number and gives me a matrix (you also need to supply an inverse, but if it is obvious enough then I can let it slide), and somewhere in your proof you need to show me that this map respects the structure: i.e. you need $f(x+y) = f(x) + f(y)$ and $f(xy) = f(x)f(y)$. Convince yourself that this is true. □

Problem 3. (from Ahlfors p.11, #1). Prove that

$$\left|\frac{a-b}{1-\bar{a}b}\right| < 1$$

if $|a| < 1$ and $|b| < 1$.

Solution. Again, basically everyone got this one. □

Y.Z.'s notes. You should always start with true statements and get to your final statement. Or you may start with your final statement and write everything in terms of equivalences - just don't lose any information on the way! □

Problem 4. (from Ahlfors p.11, #3). If $|a_i| < 1$ and $\lambda_i \geq 0$ for $i = 1, \dots, n$ and $\lambda_1 + \lambda_2 + \dots + \lambda_n = 1$, show that

$$|\lambda_1 a_1 + \lambda_2 a_2 + \dots + \lambda_n a_n| < 1.$$

Solution. Everyone got this one again - straightforward triangle inequality. \square

Y.Z.'s notes. Some people just looked up the triangle inequality - convince yourself that it is true! In particular, Cauchy's inequality gives it to you. \square

Problem 5. (from Ahlfors p.15, #2). Prove that the points a_1, a_2, a_3 of an equilateral triangle if and only if $a_1^2 + a_2^2 + a_3^2 = a_1a_2 + a_2a_3 + a_3a_1$.

Solution. The following two statements are equivalent, but tricky:

$$a_1^2 + a_2^2 + a_3^2 = a_1a_2 + a_2a_3 + a_3a_1 \tag{1}$$

$$(a_1 - a_2)^2 = (a_3 - a_2)(a_1 - a_3) \tag{2}$$

$$\tag{3}$$

Now, this gives us that $x^2 = yz, y^2 = zx, z^2 = xy$ where x, y, z are the differences between pairs of a_i . Note that we immediately get $|x|^2 = |yz|, |y|^2 = |zx|, |z|^2 = |xy|$. Add the three and factor to get $(|x| - |y|)^2 + (|y| - |z|)^2 + (|z| - |x|)^2 = 0$, which means $|x| = |y| = |z|$, as desired (since this is what it means to be equilateral).

For the other direction, it is not hard to check that if the triangle is equilateral, then x, y, z have equal magnitudes and are rotations of $e^{\pm 2\pi i/3}$ from each other. Therefore, $x, y, z = r, re^{2\pi i/3}, re^{-2\pi i/3}$ in some order, so $x^2 = yz, y^2 = zx, z^2 = xy$ is satisfied (why?), and we can get back to the algebraic equality. \square

Y.Z.'s notes. You must prove both directions for it to be an if-and-only-if! Many people did not do this. Also, most of the solutions you guys wrote for this were very long. This is the cleanest solution I know. \square

Problem 6. (from Ahlfors p.16, #1). (a) Express $\cos 3\varphi, \cos 4\varphi,$ and $\sin 5\varphi$ in terms of $\cos \varphi$ and $\sin \varphi$ and (b) express $\tan 7\varphi$ in terms of $\tan \varphi$.

Solution. Everyone got this one too. The only trouble arose with the tangent - you just needed to do both $\sin(7\phi)$ and $\cos(7\phi)$ and divide. You should get some uber-ugly mess

$$\frac{7t - 35t^3 + 21t^5 - t^7}{1 - 21t^2 + 35t^4 - 7t^6}$$

where $t = \tan(\phi)$. \square

Y.Z.'s notes. Feels weirdly like busywork. Sorry about this problem. But this technique is good to memorize! \square

Problem 7. (from Ahlfors p.16, #2). Simplify $1 + \cos \varphi + \cos 2\varphi + \dots + \cos n\varphi$ and $\sin \varphi + \sin 2\varphi + \dots + \sin n\varphi$.

Solution. Recognize these two as the real and complex parts of

$$1 + e^{i\phi} + e^{2i\phi} + \dots + e^{ni\phi} = \frac{e^{(n+1)i\phi} - 1}{e^{i\phi} - 1} \quad (4)$$

$$= \frac{e^{(2n+1)i\phi/2} - e^{-i\phi/2}}{e^{i\phi/2} - e^{-i\phi/2}} \quad (5)$$

$$= \frac{(e^{ni\phi/2})(e^{(n+1)i\phi/2} - e^{-(n+1)i\phi/2})}{2i \sin(\phi/2)} \quad (6)$$

$$= \frac{(e^{ni\phi/2})(2i \sin((n+1)\phi/2))}{2i \sin(\phi/2)} \quad (7)$$

$$= (\cos(ni\phi/2) + i \sin(ni\phi/2))e^{ni\phi/2} \frac{\sin((n+1)\phi/2)}{\sin(\phi/2)} \quad (8)$$

$$(9)$$

so you should get exactly $\cos(ni\phi/2) \sin((n+1)\phi/2) \csc(\phi/2)$ and $\sin(ni\phi/2) \sin((n+1)\phi/2) \csc(\phi/2)$, respectively. □

Y.Z.'s notes. A lot of people did some really long sums twice here, one for each part- I think it is better to think of the two as the same problem, so I present this solution. This kind of technique is clever - you see it a lot in math competitions and similar frivolous matters, though they do appear in calculations you'll see in "real life" mathematics and physics, especially Fourier analysis, which we should get into soon. □

Problem 8. (from Ahlfors p.32, #2). If Q is a polynomial with complex coefficients whose roots $\alpha_1, \dots, \alpha_n$ are all distinct, and if P is a polynomial of degree $< n$ with complex coefficients, show that

$$\frac{P(z)}{Q(z)} = \sum_{k=1}^n \frac{P(\alpha_k)}{Q'(\alpha_k)(z - \alpha_k)},$$

where $Q'(z)$ is the derivative of $Q(z)$.

Solution. Everyone got this? □

Y.Z.'s notes. □

Problem 9. (from Ahlfors p.32, #3). Use the formula in the preceding problem to prove that there exists a unique polynomial P of degree $< n$ with given values c_k at the points α_k (Lagrange's interpolation polynomial).

Solution. Again, everyone got this. However, many of you did not even mention the existence and only proved uniqueness. The problem asks for both facts. □

Y.Z.'s notes. □

Problem 10. Use the method of Cauchy's theorem (as presented in class) to establish the following three integration formulae.

$$(a) \quad \int_{-\pi}^{\pi} \frac{d\theta}{1 + \sin^2 \theta} = \sqrt{2} \pi.$$

$$(b) \quad \int_0^{2\pi} \frac{\cos^2 3\theta d\theta}{5 - 4 \cos 2\theta} = \frac{3\pi}{8}.$$

$$(c) \quad \int_0^{\pi} \sin^{2n} \theta d\theta = \frac{(2n)!}{2^{2n} (n!)^2} \pi.$$

Solution. a) Use the recipe in class, and you'd realize you need to take the integral around the unit circle of

$$\frac{4izdz}{z^4 - 6z^2 + 1}.$$

This has four roots, $\pm 1 \pm \sqrt{2}$. We obviously only care about two of them. Using partial fractions, we get that the only factors we care about are

$$\frac{4izdz}{z^4 - 6z^2 + 1} = 4i \frac{A}{z - (-1 + \sqrt{2})} + 4i \frac{B}{z - (1 - \sqrt{2})} + \text{blah}.$$

Solving for A and B , we get $-1/(8\sqrt{2})$ for both. So we get in total (remember to pick up $2\pi i$ for both!) $4i * (4\pi i)/(8\sqrt{2}) = \sqrt{2}\pi$, as desired.

b) As you know, the Professor made a mistake and thought this was some nonzero value. Many of you guys applied the techniques taught in class and got the zero value. However, you could also have reasoned it this way:

$$\int_0^{2\pi} \frac{\cos(\theta)^3 d\theta}{5 - 4 \cos(\theta)^2} = \int_{-\pi/2}^{\pi/2} \frac{\cos(\theta)^3 d\theta}{5 - 4 \cos(\theta)^2} + \int_{\pi/2}^{3\pi/2} \frac{\cos(\theta)^3 d\theta}{5 - 4 \cos(\theta)^2},$$

Now note that the cosine function is anti-symmetric across $\pi/2$ ($\cos(x) = -\cos(\pi - x)$), so the above integral is zero.

c)

$$\int_0^{\pi} \sin(\theta)^{2n} d\theta = (1/2) \int_{|z|=1} (1/2i)^{2n} (z - 1/z)^{2n} (dz/iz)$$

Some of you got stuck here. Now, keep in mind what that expansion is - we know that $(z - 1/z)^{2n}$ is just going to be some summation of things that look like z^k (where k could be zero or negative). However, we also know that except for the term where $k = 0$, everything else is going to die when we integrate (keep in mind there is a factor of $1/z$ outside!).

Therefore, it suffices to get that term where $k = 0$, and we can do this with the binomial theorem, which tells us that the $x^k y^{n-k}$ term in $(x + y)^n$ has coefficient $\binom{n}{k}$.

So, the term we want is $z^n (-1/z)^n$, and it has coefficient $\binom{2n}{n}$. Multiply by the other coefficients, we get a coefficient of

$$\frac{(-1)^n \binom{2n}{n}}{2(2i)^{2n} i} = \frac{(i)^{2n} \binom{2n}{n}}{2(2i)^{2n} i} \tag{10}$$

$$= \frac{\binom{2n}{n}}{2^2 2^{2n} i} \tag{11}$$

$$= \frac{(2n)!}{2(n!)^2 2^{2n} i} \tag{12}$$

$$\tag{13}$$

multiplying by $2\pi i$, we get the desired result. □

Y.Z.'s notes. I know this is a long set, but all three problems are necessary and demonstrate important technique. (a) is a bread-and-butter review of the integration technique that allows us to integrate all of these functions. (c) reminds us to think outside the box and just focus on the term with total power $1/z$, which is exactly what the Residue Theorem is. As for (b), well - it tells us to not mindlessly apply things we learn in class. □