Name: **SOLUTIONS**

In Class MidTerm Exam for Math 113

10 - 11:30 am, March 20, 2008

Problem	Points	Score
1	15	
2	15	
3	14	
4	15	
5	12	
6	14	
7	15	
Total	100	

- Please show **ALL** your work on this exam paper. Partial credit will be awarded where appropriate.
- Some problems will be asking you to prove results covered in class. For those that do not, you may quote without proof the theorems/propositions/lemmas given during lecture as long as you state each result clearly. Points may be deducted for incorrect or missing hypotheses.
- **NO** calculators are permitted.

1. (15 points) Simplify the following expressions:

   (a) $e^{\log i}$

   (b) $\log i$

   (c) $i^{\log(-1)}$

   SOLUTIONS: This was Exercise 2.R.13.

   (a) $e^{\log i} = i$

   (b) $\log i = i\frac{\pi}{2} + i2\pi n = i\frac{\pi}{2}(1+4n), \quad n = 0, \pm 1, \pm 2, \ldots$

   (c) $i^{\log(-1)} = e^{\log(-1)\log i} = e^{i\pi\{i\frac{\pi}{2}(1+4n)\}}, = e^{-\frac{\pi^2}{2}(1+4n)}, \quad n = 0, \pm 1, \pm 2, \ldots$

2. (15 points)

   (a) Prove that the n nth roots of unity can be expressed as $1, w, w^2, w^3, \ldots, w^{n-1}$.
   (b) What is w?
   (c) Show that the sum of the n nth roots of unity is zero.

SOLUTIONS: Parts (a) and (b) were Exercise 1.3.27.

   (a) We are looking for solutions to the equation

$$1^{1/n} = w.$$

   Equivalently $w^n = 1$. It is known from elementary algebra that this equation has at most n distinct solutions.
   Define $w = e^{2\pi i/n}$. We notice that for any $k = 0, \pm 1, \pm 2, \ldots$

$$(w^k)^n = (e^{2\pi ik/n})^n = e^{2\pi ik} = 1.$$

   However $1 = w^0, w, w^2, w^3, \ldots, w^{n-1}$ are distinct since $w^k = w^j$ implies that $e^{2\pi i(k-j)/n} = 1$ which implies $k = j \mod n$.

   (b) $w = e^{2\pi i/n}$.

   (c) The sum of the roots is

$$1 + w + \cdots + w^{n-1}$$

   Notice $w^n - 1 = 0$. And $w^n - 1 = (w - 1)(1 + w + \cdots + w^{n-1})$. Either $n = 1$ or $n > 1$ and $w \neq 1$. In both cases we achieve the desired conclusion.

3. (a) (7 points) Is $f(z) = \overline{z}$ analytic? Prove or disprove.

   (b) (7 points) Let A be an open subset of $\mathcal{C}$ and $A^* = \{z : \overline{z} \in A\}$. Suppose f is analytic on A and define a function g on A^* by

$$g(z) = \overline{f(\overline{z})}.$$

   Show that g is analytic on A^*.

SOLUTIONS: This was Example 1.5.17 and Example 1.5.19.

(a) The function does not satisfy the Cauchy-Riemann equations.

(b) The proof is on page 75 of Marsden and Hoffman.

4. (15 points)

   (a) Define what it means for a function to be analytic (holomorphic) at a point $z \in \mathbb{C}$.

   (b) State the Cauchy-Riemann equations.

   (c) Prove that a function analytic at a point $z \in \mathbb{C}$ satisfies the Cauchy-Riemann equations at that point.

   SOLUTIONS:

   (a) See Definition 1.5.1 in Marsden and Hoffman.

   (b) $\partial u/\partial x = \partial v/\partial y$, $\partial u/\partial y = -\partial v/\partial x$

   (c) See the first half of the proof of Theorem 1.5.8.

5. Evaluate the following integrals with γ being the unit circle centered at the origin:

(a) (3 points) $\int_\gamma \sin z \; dz$.

(b) (3 points) $\int_\gamma \frac{\sin z}{z} \; dz$.

(c) (3 points) $\int_\gamma \frac{\sin z}{z^2} \; dz$.

(d) (3 points) $\int_\gamma \frac{\sin(e^z)}{z^2} \; dz$.

SOLUTIONS: This was Exercise 2.R.1.

(a) (3 points) $\int_\gamma \sin z \; dz = 0$. (Cauchy's Thm)

(b) (3 points) $\int_\gamma \frac{\sin z}{z} \; dz = 2\pi i \sin 0 = 0$. (Cauchy's Integral formula)

(c) (3 points) $\int_\gamma \frac{\sin z}{z^2} \; dz = \frac{2\pi i}{1!} \cos 0 = 2\pi i$. (Cauchy's Integral formula)

(d) (3 points) $\int_\gamma \frac{\sin(e^z)}{z^2} \; dz = 2\pi i \cos(e^0) = 2\pi i \cos(1)$. (Cauchy's Integral formula)

6. (14 points) Evaluate $\int_0^{2\pi} e^{-i\theta} e^{e^{i\theta}} d\theta$.

SOLUTIONS: This was Exercse 2.R.11. We have $\int_0^{2\pi} e^{-i\theta} e^{e^{i\theta}} d\theta = 2\pi$.

The way to see this is to reverse engineer Cauchy's integral formula. Consider $\gamma = \{z : |z| = 1\}$. Then

$$\frac{1!}{2\pi i} \int_\gamma \frac{e^z}{z^2} dz = I(0, \gamma) \left. \frac{de^z}{dz} \right|_{z=0} = 1,$$

which is just Cauchy's integral formula for derivatives.

On the other hand, with $z = e^{i\theta}$ we have the explicit integration

$$\int_\gamma \frac{e^z}{z^2} dz = \int_0^{2\pi} \frac{e^{e^{i\theta}}}{e^{2i\theta}} i e^{i\theta} d\theta = i \int_0^{2\pi} e^{e^{i\theta}} e^{-i\theta} d\theta.$$

Putting these together yields the result.

7. (15 points) Let f be entire and let $|f(z)| \leq M$ for z on the circle $|z| = R$ with R fixed. Prove that

$$\left|f^{(k)}(re^{i\theta})\right| \leq \frac{k!M}{(R-r)^k},$$

for all $0 \leq r < R$ and for all $k = 0, 1, 2, 3, \ldots$

SOLUTIONS: This was Exercise 2.R.8. It is a nice generalization of Cauchy's inequality.

Let $z_0 = re^{i\theta}$ and define the curve

$$\gamma = \{z : |z - z_0| = R - r\}.$$

Notice that $\gamma \subset \{z : |z| \leq R\}$. Therefore by the maximum modulus Principle $|f| \leq M$ on γ.

The Cauchy integral formula gives us

$$f^{(k)}(z_0) = \frac{k!}{2\pi i}\int_\gamma \frac{f(\zeta)}{(\zeta - z_0)^{k+1}}d\zeta.$$

Now, since z_0 is the center of γ, Cauchy's inequality grants us the conclusion:

$$\left|f^{(k)}(re^{i\theta})\right| \leq \frac{k!M}{(R-r)^k}.$$

That proof uses the maximum modulus principle.

Extra space for work: