Problem 1 (§3.2, 14). Let $\sum b_n(z - z_0)$ be a Taylor expansion of $f(z)$ about z_0. If $|z_1 - z_0| < R - |z_0|$, then by the triangle inequality,

$$|0 - z_1| \le |0 - z_0| + |z_0 - z_1| < |z_0| + R - |z_0| = R,$$

which implies by definition of the radius of convergence that $\sum a_n z^n$ converges at z_1. Therefore, $\sum a_n z^n$ converges on the entire disc $D(z_0, R - |z_0|)$; by Taylor's theorem, $\sum b_n z^n$ converges everywhere on $D(z_0, R - |z_0|)$ as well. Therefore, $R - |z_0| \le \widetilde{R}$.

Note: It is not true that if $\sum a_n z^n$ converges at z_1, then $\sum b_n(z - z_0)^n$ automatically converges at z_1 as well. For a counterexample, consider expansions of the function $1/(1 - z)$ about $z = 0$ and $z = 2$.

If we suppose that $\widetilde{R} > R + |z_0|$, then there exists $\epsilon > 0$ for which $\widetilde{R} > R + |z_0| + \epsilon$. If $|z_1| = R + \epsilon$, then by the triangle inequality,

$$|z_0 - z_1| \le |z_0| + |z_1| = R + |z_0| + \epsilon < \widetilde{R},$$

which implies that $\sum b_n(z - z_0)$ converges on the entire disc $D(0, R + \epsilon)$. By Taylor's theorem, $\sum a_n z^n$ must converge on all of $D(0, R + \epsilon)$ as well, which contradicts the fact that R is the radius of convergence of $\sum a_n z^n$.Therefore, $\widetilde{R} \le R + |z_0|$.

Problem 2 (§3.2, 16). By Taylor's theorem $f(z)$ is analytic on A. Therefore, since A is simply connected, $\int_\gamma f = 0$ by Cauchy's theorem.

Problem 3 (§3.2, 20). Given any z_1, z_2 on the circle of convergence of $\sum a_n z^n$, we know by definition that

$$\sum |a_n z_1^n| = \sum |a_n z_2^n| = \sum |a_n| R^n,$$

where R is the radius of convergence. In particular,

$$\sum |a_n z_1^n| < \infty \iff \sum |a_n z_2^n| < \infty.$$

Therefore, $\sum a_n z^n$ converges everywhere or nowhere on its circle of convergence.

The series $\sum z^n/n$ and $\sum z^n/n^2$ both have radius of convergence 1, as was checked on a previous problem set. However, since $\sum 1^n/n = \infty$ and $\sum 1^n/n^2 = \pi^2/6$, $\sum z^n/n^2$ is these only one of the two series that converges on its circle of convergence. It is noteworthy that a series converges on its circle of convergence iff it converges uniformly on the open disc $D(z_0, R)$.

Problem 4 (§3.3, 1). a) The Taylor series for $\sin z$ is

$$\sin z = \sum_{n=0}^{\infty} \frac{(-1)^n z^{2n+1}}{(2n+1)!},$$

and this converges on the entire plane. We can there for substitute in $1/z$ where ever this is defined (which is for $0 < |z| < \infty$ to obtain

$$\sin \frac{1}{z} = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)! z^{2n+1}} = \frac{1}{z} - \frac{1}{3! z^3} + \frac{1}{5! z^5} - \cdots.$$

b) For $0 < |z| < 1$ we have the gemetric series expansion

$$\frac{1}{z+1} = \sum_{n=0}^{\infty}(-1)^n z^n = 1 - z + z^2 - z^3 + \cdots .$$

Therefore, on this region we obtain

$$\frac{1}{z(z+1)} = \sum_{n=0}^{\infty} \frac{(-1)^n z^n}{z} = \sum_{n=-1}^{\infty} (-1)^{n+1} z^n = \frac{1}{z} - 1 + z - z^2 + \cdots .$$

c) Using the same geometric expansion (which is valid on $0 < |z| < 1$), we multiply through by z to obtain

$$\frac{z}{z+1} = z\sum_{n=0}^{\infty}(-1)^n z^n = \sum_{n=0}^{\infty}(-1)^{n+1} z^n = z - z^2 + z^3 - \cdots .$$

d) We have

$$e^z = \sum_{n=0}^{\infty} \frac{z^n}{n!},$$

which converges on the entire plane. The one-term Laurent series $1/z^2$ converges on $0 < |z| < \infty$, so the product of the two converges on this same region and is given by

$$\frac{e^z}{z^2} = \sum_{n=0}^{\infty} \frac{z^{n-2}}{n!} = \sum_{n=-2}^{\infty} \frac{z^n}{(n+2)!} = \frac{1}{z^2} + \frac{1}{z} + \frac{1}{2!} + \frac{z}{3!} + \cdots .$$

Problem 5 (§3.3, 5). Let γ_1 and γ_2 be two concentric circles around z_0 of radii R_1 and R_2 respectively, with $R_1 < R_2$. Suppose that z lies between the circles and that f is analytic on a region containing both curves and the enclosed annulus. We will show that

$$f(z) = \frac{1}{2\pi i}\int_{\gamma_2} \frac{f(\zeta)}{\zeta - z}\, d\zeta - \frac{1}{2\pi i}\int_{\gamma_1} \frac{f(\zeta)}{\zeta - z}\, d\zeta.$$

Indeed, let γ_3 be a radial segment from γ_1 to γ_2. Let $\gamma = \gamma_2 + \gamma_3 - \gamma_1 - \gamma_3$ be the composite path which loops around γ_2, moves along γ_3 to γ_1, goes around, and then follows γ_3 back to the starting point (there's a picture of this path on page 492 in the solutions in the book). Let γ_4 be a small circle around z which lies between γ_1 and γ_2. It is clear that γ is homotopic in the region of analyticity to γ_4 and that the winding number of γ_4 around z_0 is 1. Thus we apply the Cauchy integral formula to obtain

$$\begin{aligned} f(z) &= \frac{1}{2\pi i}\int_{\gamma_4} \frac{f(\zeta)}{\zeta - z}\, d\zeta = \frac{1}{2\pi i} \frac{f(\zeta)}{\zeta - z}\, d\zeta \\ &= \frac{1}{2\pi i}\int_{\gamma_2} \frac{f(\zeta)}{\zeta - z}\, d\zeta + \frac{1}{2\pi i}\int_{\gamma_3} \frac{f(\zeta)}{\zeta - z}\, d\zeta - \frac{1}{2\pi i}\int_{\gamma_1} \frac{f(\zeta)}{\zeta - z}\, d\zeta - \frac{1}{2\pi i}\int_{\gamma_3} \frac{f(\zeta)}{\zeta - z}\, d\zeta \\ &= \frac{1}{2\pi i}\int_{\gamma_2} \frac{f(\zeta)}{\zeta - z}\, d\zeta - \frac{1}{2\pi i}\int_{\gamma_1} \frac{f(\zeta)}{\zeta - z}\, d\zeta, \end{aligned}$$

as desired.

Problem 6 (§3.3, 8). We'll prove the complex version of L'Hôpital's rule. Let f and g be analytic on a region containing z_0 and each have zeroes of order k at z_0. It follows by Proposition 3.3.5 that f/g has a removable singularity at z_0. Since the functions are analytic and have the first k coefficients vanishing, they have Taylor expansions

$$f(z) = \sum_{n=k}^{\infty} a_n(z-z_0)^n = (z-z_0)^k \phi(z)$$
$$g(z) = \sum_{n=k}^{\infty} b_n(z-z_0)^n = (z-z_0)^k \psi(z),$$

where $\phi(z_0) = a_k = f^{(k)}(z_0)/k! \neq 0$ and $\psi(z) = b_k = g^{(k)}(z_0)/k! \neq 0$, since the order of each is zero is exactly k by assumption. Then, using the fact that both ϕ and ψ have nonzero limit as z approaches z_0,

$$\lim_{z\to z_0} \frac{f(z)}{g(z)} = \lim_{z\to z_0} \frac{(z-z_0)^k\phi(z)}{(z-z_0)^k\psi(z)} = \lim_{z\to z_0} \frac{\phi(z)}{\psi(z)} = \frac{\phi(z_0)}{\psi(z_0)} = \frac{f^{(k)}(z_0)}{f^{(k)}(z_0)}.$$

Problem 7 (§3.3, 15). Suppose that the series

$$\sum_{n=1}^{\infty} \frac{b_n}{(z-z_0)^n}$$

converges for $|z-z_0| > R$. We claim that it converges uniformly on $F_r = \{z \in \mathbb{C} : |z-z_0| > r\}$ for any $r > R$.

Fix s so $R < s < r$. The book (and unsurprisingly most of the students) take $s = (r+R)/2$, the most obvious choice. By assumption, $\sum b_n/(z-z_0)^n$ converges on $|z-z_0| = s > R$. It particular, this implies that the terms $b_n/(z-z_0)^2$ tend to 0 on $|z-z_0| = s$, so b_n/s^n go to 0 and in particular are bounded. It follows by Abel-Weierstrass that $\sum b_n(z-z_0)^n$ converges absolutely and uniformly on the disk $D_{1/r,z_0}$ of radius $1/r < s$. Thus $\sum |b_n|/r^n$ converges and for $z \in F_r$, we have $|z-z_0| > r$, so

$$\left|\frac{b_n}{(z-z_0)^n}\right| < \frac{|b_n|}{r^n}.$$

It follows by the M test that our series converges uniformly on F_r.