Math 113 Assignment 7 Solutions

Problem 1 (§3.1, 1). a) This sequence does not converge. A convergent sequence is a Cauchy sequence, but

$$\lim_{n\to\infty} |z_n - z_{n+1}| = |(-1)^n - (-1)^{n+1}| = 2.$$

b) Since

$$\lim_{n\to\infty} \left|\frac{n!}{n^n} i^n\right| = \lim_{n\to\infty} \left(\frac{1}{n}\right)\left(\frac{2}{n}\right)\cdots\left(\frac{n}{n}\right) < \lim_{n\to\infty} \frac{1}{n} = 0,$$

z_n converges to zero.

Problem 2 (§3.1, 4). a) Fix $z \in A := \mathbb{C} \setminus \{z = ni \mid n \text{ is an integer}\}$. Since

$$\lim_{n\to\infty} \frac{1/n^2}{1/|n^2+z^2|} = 1,$$

we know by the limit comparison test that $\sum 1/n^2$ and $\sum 1/|n^2+z^2|$ either both converge or both diverge. $\sum 1/n^2$ is obviously a convergent p-series; thus, $\sum 1/(n^2+z^2)$ converges as well.

b) We already proved that $\sum 1/(n^2+z^2)$ converges absolutely. In addition, given any compact set $K \subset A$, there exists $z_0 \in K$ with minimal absolute value. Evidently, $|z| \geq |z_0| \Rightarrow 1/|n^2+z^2| \leq 1/|n^2+z_0|$. By point-wise convergence, there exists $N > 0$ for which

$$\sum_{n=N}^{\infty} \frac{1}{|n^2+z_0^2|} < \epsilon.$$

Term-by-term comparison tells us that

$$\sum_{n=N}^{\infty} \frac{1}{|n^2+z^2|} \leq \sum_{n=N}^{\infty} \frac{1}{|n^2+z_0^2|} < \epsilon$$

for all $z \in K$. Therefore, $\sum 1/(n^2+z^2)$ converges uniformly on every compact subset of A.

Problem 3 (3.1.7). a) We claim that the sum

$$\sum_{n=2}^{\infty} \frac{i^n}{\log n}$$

converges but not absolutely. To test, absoluteness, not that $\log n < n$ for $n > 2$, and so $1/\log n > 1/n$. It follows that

$$\sum_{n=2}^{\infty} \left|\frac{i^n}{\log n}\right| = \sum_{n=2}^{\infty} \frac{1}{\log n}$$

does not converge by comparison to the harmonic series, and so the series does not converge absolutely. On the other hand, we claim that it does converge. It suffices to show that the partial sums have a limit. We have

$$S_M = \sum_{n=2}^{M} \frac{i^n}{\log n} = \sum_{\substack{n\geq 2\\ n \text{ even}}} \frac{(-1)^{n/2}}{\log n} + i \sum_{\substack{n\geq 3\\ n \text{ odd}}} \frac{(-1)^{(n+1)/2}}{\log n}$$

Both the real and imaginary parts of the partial sums have limits by the alternating series test from real analysis. It follows by an old homework problem that the partial sums S_M themselves converge, and so the series is convergent.

b) Consider now

$$\sum_{n=2}^{\infty} \frac{i^n}{n}.$$

We claim that this too converges but not absolutely. The failure of absolute converges is immediate from the fact that $|i^n/n| = 1/n$ and comparison with the harmonic series. The same method works as before to show convergence. The partial sums are given by

$$S_M = \sum_{n=2}^{M} \frac{i^n}{n} = \sum_{\substack{n \geq 2 \\ n \text{ even}}} \frac{(-1)^{n/2}}{n} + i \sum_{\substack{n \geq 3 \\ n \text{ odd}}} \frac{(-1)^{(n+1)/2}}{n},$$

and both the real and imaginary parts converge by the alternating series test.

Problem 4 (3.1.9). Suppose that $\sum_{k=1}^{\infty} g_k(z)$ is a uniformly convergent series of continuous functions. Then $g(z) = \sum_{k=1}^{\infty} g_k(z)$ is continuous by Proposition 3.1.6. Then if $z_n \to z$, applying continuiting of g we have

$$\lim_{n\to\infty} \sum_{k=1}^{\infty} g_k(z_n) = \lim_{n \to infty} g(z_n) = g(z) = \sum_{k=1}^{\infty} g_k(z),$$

as claimed.