Math 113 Assignment 6 Solutions

Problem 1 (§2.3, 1). We claim that $A = \mathbb{C} \setminus \{0\}$ is not simply connected. Note that $1/z$ is an analytic function on A, and let $\gamma(t) = e^{it}$ for $t \in [0, 2\pi]$. We have seen that

$$\int_\gamma \frac{1}{z}\, dz = 2\pi i,$$

and it follows that γ is not nulhomotopic (that is, homotopic to a point), since if it were, this integral would be 0 by Cauchy's Theorem 2.3.14. Thus A contains a loop which is not nulhomotopic and so is not simply connected.

Some topologists among you remarked that A deformation retracts to the unit circle and thus has the homotopy type of S^1. This also works, but probably isn't the intended solution.

Problem 2 (§2.3, 3). Say that a region A is *star-shaped with respect to* z_0 if for any $z \in A$ and $0 \leq s \leq 1$ we have $sz_0 + (1-s)z \in A$.

We claim that a star-shaped set is simply connected; to show this we will demonstrate that every loop is homotopic to a point. Let A be star-shaped with respect to z_0 and let $\gamma : [a, b] \to A$ be a closed curve in A, so $\gamma(a) = \gamma(b)$. We could assume that $a = 0$ and $b = 1$ by reparametrizing, but there's no need. Define a homotopy $H : [0, 1] \times [a, b] \to A$ by

$$H(s, t) = s\gamma(t) + (1 - s)z_0.$$

This is continuous, since it's a sum and product of continuous functions. It maps A into A by the definition of star-shaped given in the problem. Moreover, H satisfies

$$\begin{aligned} H(0,t) &= 0 \cdot \gamma(t) + 1z_0 = z_0, & H(s,a) &= s\gamma(a) + (1-s)z_0 \\ H(1,t) &= 1 \cdot \gamma(t) + 0 \cdot z_0 = \gamma(t), & H(s,b) &= s\gamma(b) + (1-s)z_0. \end{aligned}$$

Since $\gamma(a) = \gamma(b)$, the endpoints at time s coincide for all $s \in [0, 1]$. So H is a homotopy between γ and z_0 (see definition 2.3.7). Note that this is a homotopy between closed curves, but not a homotopy with fixed endpoints (see Definition 2.3.6 and Definition 2.3.7); the point z_0 might not even lie on γ.

The idea of this homotopy is that given a loop, every point on the loop can be moved along a straight line to the base z_0 without leaving A, since it is star-shaped, so the whole loop is homotopic to a point.

Problem 3 (§2.3, 4). We claim that A is convex if and only if it is star-shaped with respect to each of its points.

Suppose first that A is star-shaped with respect to each of its points. We need to check that for any $z_0, z_1 \in A$, $sz_1 + (1-s)z_0 \in A$ (this is Definition 2.3.9). This follows from the definition of star-shaped, since A is star-shaped with respect to z_0 and we can take "z" in the definition to be z_1.

The other direction is similar. Suppose that A is convex and fix z_0 in A. For any other $z \in A$, by the definition of convexity we have $sz + (1-s)z_0 \in A$, and so A satisfies the condition of being star-shaped with respect to z_0.

Problem 4 (§2.3, 7). a) The function $1/z$ is analytic on $A = \mathbb{C} \setminus \{0\}$. We claim that $\gamma(t) = \cos t + 2i \sin t$ ($0 \leq t \leq 2\pi$) is homotopic to $\gamma_0(t) = e^{it}$. Define a homotopy $H : [0, 1] \times [0, 2\pi] \to A$ by

$$H(s, t) = \cos t + (1 + s)i \sin t.$$

$H(s,t) \neq 0$ for all pairs (s,t), so this is indeed a homotopy in A since it is also seen to fix endpoints. Then $H(0,t) = \cos t + i \sin t = e^{it} = \gamma_0(t)$, and $H(1,t) = \cos t + i \sin 2t = \gamma(t)$, so γ and γ_0 are homotopic. It follows then by Cauchy's theorem that

$$\int_\gamma \frac{1}{z}\, dz = \int_{\gamma_0} \frac{1}{z}\, dz = 2\pi i,$$

as we have seen before.

b) Applying Cauchy's theorem again to the same paths,

$$\int_\gamma \frac{1}{z^2}\, dz = \int_{\gamma_0} \frac{1}{z^2}\, dz = 0.$$

c) Set $\gamma(t) = 2 + e^{it}$ $(0 \leq t \leq 2\pi)$. I claim that γ is nulhomotopic in A. Define $H : [0,1] \times [0, 2\pi] \to A$ by

$$H(s,t) = 2se^{it} + 1.$$

Then $H(0,t) = 1$ is a trivial path, and $H(1,t) = 2e^{it} + 1 = \gamma(t)$. So γ is nulhomotopic and by Cauchy's theorem,

$$\int_\gamma \frac{e^z}{z}\, dz = 0.$$

d) Let $\gamma(t) = 1 + e^{it}$, $(t \in [0, 2\pi])$. Note that

$$\frac{1}{z^2-1} = \frac{1/2}{z-1} - \frac{1/2}{z+1},$$

and so

$$\int_\gamma \frac{1}{z^2-1} = \frac{1}{2}\int_\gamma \frac{1}{z-1} - \frac{1}{2}\int_\gamma \frac{1}{z+1} = \frac{1}{2}(2\pi i) + 0 = \pi i.$$

The first integral is equal to $\int_{|z|=1} dz/z = 2\pi i$, and the second is 0, since γ is homotopic to a point in $\mathbb{C}\setminus\{-1\}$ (the domain of analyticity of $1/(z+1)$ by the homotopy $H(s,t) = 1 + se^{it}$.

Problem 5 (§2.4, 6). Let f be analytic on a region A and let γ be a closed curve in A. Fix $z_0 \in A$ not on γ (so $\gamma(t) \neq z_0$ for all t). By the Cauchy Integral Formula 2.4.6, for $k = 1, 2, 3, \ldots$, we have

$$f^{(k)}(z_0) \cdot I(\gamma; z_0) = \frac{k!}{2\pi i}\int_\gamma \frac{f(\zeta)}{(\zeta - z_0)^{k+1}}\, d\zeta.$$

Since f is analytic on A, it follows by Theorem 2.4.6 that $f^{(k)}$ is analytic on A and then satisfies the conditions of Theorem 2.4.4. Applying that theorem to $f^{(k)}$ yields

$$f^{(k)}(z_0) \cdot I(\gamma; z_0) = \frac{1}{2\pi i}\int_\gamma \frac{f^{(k)}(\zeta)}{\zeta - z_0}\, d\zeta.$$

Equating these and cancelling the factor of $2\pi i$ yields

$$\int_\gamma \frac{f^{(k)}(\zeta)}{\zeta - z_0}\, d\zeta = k!\int_\gamma \frac{f(\zeta)}{(\zeta - z_0)^{k+1}}\, d\zeta.$$

This is the generalization desired; in the case $k = 1$, this specializes to

$$\int_\gamma \frac{f'(\zeta)}{\zeta - z_0}\, d\zeta = \int_\gamma \frac{f(\zeta)}{(\zeta - z_0)^2}\, d\zeta,$$

the result stated in the problem.

Problem 6 (§2.4, 11). Suppose that F is analytic on A. Fix $z_0 \in A$ and define f on A by

$$f(z) = \begin{cases} \frac{F(z)-F(z_0)}{z-z_0} & \text{if } z \neq z_0, \\ F'(z_0) & \text{if } z = z_0. \end{cases}$$

It follows immediately from Proposition 1.5.3 that f is analytic everywhere on A except possibly at z_0, since it is a quotient of two analytic functions and the denominator in nonvanishing, so we need only check that f is analytic at z_0. Since F is analytic, the limit

$$\lim_{z \to z_0} f(z) = \lim_{z \to z_0} \frac{F(z) - F(z_0)}{z - z_0} = F'(z_0) = f(z_0)$$

exists, so f is continuous at z_0 and thus on all of A. It follows from Morera's theorem (more properly, Corollary 2.4.11) that f is analytic on A.

Problem 7 (§2.4, 20). Note that $1/(z - z_0)$ is continuous on γ_1 since z_0 is by assumption not on γ_1. Then by the definition of the winding number and Proposition 2.1.3,

$$I(-\gamma_1; z_0) = \frac{1}{2\pi i}\int_{-\gamma_1} \frac{dz}{z - z_0} = -\frac{1}{2\pi i}\int_{\gamma_1} \frac{dz}{z - z_0} = -I(\gamma_1; z_0).$$

Geometrically, this means that if γ_1 goes around z_0 a total of n times (i.e.$I(\gamma_1; z_0) = n$), then the curve $-\gamma_1$, which is the traversal of the same path in the opposite direction, has winding number $-n$, so it goes around z_0 the same number n times, but in the opposite direction.

Similarly, if γ_1 and γ_2 are closed curves which may be added, then

$$I(\gamma_1, \gamma_2; z_0) = \int_{\gamma_1 + \gamma_2} \frac{dz}{z - z_0} = \int_{\gamma_1} \frac{dz}{z - z_0} + \int_{\gamma_2} \frac{dz}{z - z_0} = I(\gamma_1; z_0) + I(\gamma_2; z_0).$$

If γ_1 winds around z_0 m_1 times, and γ_2 goes around m_2, than their sum, which corresponds to following γ_1 and then γ_2, goes around $m_1 + m_2$ times.