Math 113 Assignment 4 Solutions

Problem 1 (§1.6, 14). By definition, there is a bijection between the branches of a multi-valued function $f : \mathbb{C} \to \mathbb{C}$ and the fundamental regions $U_i \subset \mathbb{C}$ for which $f^{-1}(U_i) = \mathbb{C} \setminus L$ for some possibly empty collection of lines L and $f : \mathbb{C} \setminus L \to U_i$ is a single-valued bijection. In general, we can partition the target space into fundamental regions in a number of different ways by making different choices of L. To pick a branch of $f(z) = \sqrt{z}$, we will follow convention and pick L to be the negative real axis so that the two branches of $\sqrt{z}$ map $\mathbb{C} \setminus \mathbb{R}_{<0}$ into the fundamental regions $U_1 = \{z \mid -\pi/2 < \arg z < \pi/2\}$ and $U_2 = \{z \mid \pi/2 < \arg z < 3\pi/2\}$. If we let $f_1 : \mathbb{C} \setminus \mathbb{R}_{<0}$ and $f_2 : \mathbb{C} \setminus \mathbb{R}_{<0}$ denote the associated branches of $\sqrt{z}$, then we can see that $0 \in f_2(\mathbb{C} \setminus \mathbb{R}_{<0}) + 1$, while $0 \notin f_1(\mathbb{C} \setminus \mathbb{R}_{<0})$. Since 0 is a branch point of $\sqrt{z}$, it is impossible to pick a branch f_i of f that is analytic on $f_2(\mathbb{C} \setminus \mathbb{R}_{<0})$. However, both f_1 and f_2 are analytic on $f_1(\mathbb{C} \setminus \mathbb{R}_{<0})$, which means that $f_1(1 + f_1(z))$ and $f_2(1 + f_1(z))$ are both analytic branches of $\sqrt{1 + \sqrt{z}}$. Interestingly enough, if we defy convention and pick $L := \mathbb{R}_{>0}$, then we obtain branches f_1' and f_2' of $\sqrt{z}$ for which all possible branches $f_i'(1 + f_j'(z))$ are analytic on $\mathbb{C} \setminus L$!

Problem 2 (Chapter 1 Review, 6). It can be verified from the definitions of $\sin z$ and $\cos z$ in terms of exponentials that standard trigonometric identities hold for complex arguments. Thus, we can use the identity $\sin^2 z + \cos^2 z = 1$ to calculate that

$$\cos z = \pm\sqrt{1 - (\sqrt{3})^2} = \pm\sqrt{-2} = \pm i\sqrt{2}.$$

We know by Euler's formula that

$$e^{iz} = \cos z + i \sin z = \pm i\sqrt{2} + i\sqrt{3},$$

which implies that

$$z = -i\left(\log\left(\pm i\sqrt{2} + i\sqrt{3}\right) + 2\pi i n\right) = -i\left(\log i + \log\left(\pm\sqrt{2} + \sqrt{3}\right)\right) + 2\pi n = -i \log\left(\pm\sqrt{2} + \sqrt{3}\right) + \frac{\pi}{2} + 2\pi i n.$$

Problem 3 (Chapter 1 Review, 10). As in problem 1.6.14, the set on which $\sqrt{z^2 - 2}$ is analytic depends on how we choose to restrict the domain of $f(z) = \sqrt{z}$. If we restrict f to $\mathbb{C} \setminus \mathbb{R}_{\leq 0}$ as usual, then we find that $\sqrt{z^2 - 2}$ is analytic on $\{z \mid z^2 - 2 \notin \mathbb{R}_{\leq 0}\} = \{z \mid z^2 \notin [2, -\infty)\} = \mathbb{C} \setminus (i\mathbb{R} \cup [-\sqrt{2}, \sqrt{2}])$.

Problem 4 (§2.1, 2). a) We'll evaluate this integral by summing over the two components of the path. Let $\gamma_1(t) = it$ for $t \in [0, 1]$ and $\gamma_2(t) = i + (t - 1)$ for $t \in [1, 3]$; then $\gamma = \gamma_1 + \gamma_2$. Then

$$\begin{aligned}\int_\gamma x\,dz = \int_\gamma \operatorname{Re} z\,dz = \int_{\gamma_1} \operatorname{Re} z\,d + \int_{\gamma_2} \operatorname{Re} z\,dz = \int_0^1 \operatorname{Re}(it)(i)\,dt + \int_1^3 \operatorname{Re}(i + (t - 1))(1)\,dt \\ = \int_0^1 0\,dt + \int_1^3 (t - 1)\,dt = 0 + \left(\frac{t^2}{2} - t\right)\Big|_1^3 = \frac{3}{2} - \left(-\frac{1}{2}\right) = 2.\end{aligned}$$

b) The function $z^2 + 2z + 3$ has an antiderivative $F(z) = \frac{z^3}{3} + z^2 + 3z$ on the entire plane. By the fundamental theorem of calculus, we can write

$$\int_\gamma (z^2 + 2z + 3)\,dz = \left(\frac{z^3}{3} + z^2 + 3z\right)\Big|_1^{2+i} = \left(\frac{29}{3} + \frac{32}{3}i\right) - \left(\frac{13}{3}\right) = \frac{16}{3} + \frac{32}{3}i.$$

c) Parametrize γ by $\gamma(t) = 1 + 2e^{it}$ for $t \in [0, 2\pi]$. Then

$$\int_\gamma \frac{1}{z-1}\,dz = \int_0^{2\pi} \frac{1}{(1+2e^{it})-1}\left(2ie^{it}\right)\,dt = \int_0^{2\pi} i\,dt = 2\pi i.$$

Problem 5 (§2.1, 4). First we expand $1/(z^2-2z)$ with partial fractions. Write

$$\frac{1}{z^2-2z} = \frac{A}{z} + \frac{B}{z-2} = \frac{Az - 2A + Bz}{z^2 - 2z},$$

whence $A = -1/2$ and $B = 1/2$. Parametrize γ by $\gamma(t) = 2 + e^{it}$ for $t \in [0, 2\pi]$. Then

$$\int_\gamma \frac{1}{z^2-2z}\,dz = \int_\gamma \frac{-1/2}{z}\,dz + \int_\gamma \frac{1/2}{z-2}\,dz.$$

Now, the left-hand summand has an antiderivative $(-1/2)\log z$ which is defined on all of γ (where we choose the branch of log omitting the positive real aaxis). So this integral is 0 by Cauchy's theorem. The second integral must be computed directly

$$\int_\gamma \frac{1/2}{z-2}\,dz = \frac{1}{2}\int_0^{2\pi} \frac{1}{(2+e^{it})-2}(ie^{it})\,dt = \frac{1}{2}\int_0^{2\pi} i\,dt = \pi i.$$

Problem 6 (§2.1, 5). It is not true that

$$\mathrm{Re}\left\{\int_\gamma f\,dz\right\} = \int_\gamma (\mathrm{Re}\,f)\,dz.$$

As a counterexample, set $f(z) = i$ and $\gamma(t) = it$ for $t \in [0, 1]$. Then

$$\mathrm{Re}\left(\int_\gamma f(z)\,dz\right) = \mathrm{Re}\left(\int_0^1 i(i)\,dt\right) = \mathrm{Re}(-1) = -1.$$

On the other hand,

$$\int_\gamma \mathrm{Re}\,f(z)\,dz = \int_\gamma 0\,dz = 0.$$