Problem 1 (§1.3, 5). a) $1 = 1 \cdot e^{i0}$, so $\log 1 = \log(1) + i(0 + 2\pi n) = 2\pi n i$, where n ranges over $\mathbb{Z}$.

b) $i = 1 \cdot e^{i\pi/2}$, so $\log i = \log(1) + i(\pi/2 + 2\pi n) = \pi i/2 + 2\pi n i$, where again $n \in \mathbb{Z}$.

Problem 2 (§1.3, 7). a) Recall that $(-i)^i = e^{i \log(-i)}$ by definition, so our first step is to find all values of $\log(-i)$. We can write $-i = 1 \cdot e^{-i\pi/2}$, and so (as in the preceding problem), $\log(-i) = 3\pi i/2 + 2\pi i n$ ($n \in \mathbb{Z}$). Then

$$(-i)^i = e^{i(-\pi i/2 + 2\pi i n)} = e^{\pi/2 - 2\pi n},$$

where n ranges over $\mathbb{Z}$. Note that we could use $-n$ instead of n, and this would range over exactly the same values, so this is equivalent to the set of $e^{\pi/2+2\pi n}$. Similarly, using $n+1$ instead of n would also range over the same values, so this could just as well be written $e^{-3\pi/2+2\pi n}$ or $e^{-3\pi/2-2\pi n}$ (with $n \in \mathbb{Z}$ in all cases).

b) The first step in finding all values of $(1+i)^{1+i}$ is to compute all values of $\log(1+i)$. We can write $1 + i = \sqrt{2} e^{i\pi/4}$ (by computing the modulus and argument), and so

$$\log(1+i) = \log\sqrt{2} + (\pi/4 + 2\pi n)i, \quad (n \in \mathbb{Z}).$$

Next we expand using the definition of exponentiation,

$$\begin{aligned}
(1+i)^{1+i} &= e^{(1+i)\log(1+i)} = e^{(1+i)(\log\sqrt{2} + (\pi/4 + 2\pi n)i)} \\
&= e^{(\log\sqrt{2} - \pi/4 - 2\pi n) + (\log\sqrt{2} + \pi/4 + 2\pi n)i} \\
&= \sqrt{2} e^{-\pi/4 - 2\pi n} e^{(\log\sqrt{2} + \pi/4)i} \\
&= \sqrt{2} e^{-\pi/4 - 2\pi n} \left(\cos\left(\frac{1}{2}\log 2 + \frac{\pi}{4}\right) + i \sin\left(\frac{1}{2}\log 2 + \frac{\pi}{4}\right) \right),
\end{aligned}$$

again with $n \in \mathbb{Z}$.

Problem 3 (§1.3, 20). a) Fix a branch of logarithm; we claim that $a^b a^c = a^{b+c}$ for complex numbers a, b, c. Since we're in a fixed branch, $\log a$ is just some constant; there's no $+2\pi n i$ term on the end. I'll write this as $\operatorname{Log} a$ to remind us of this. Now we apply the definition and properties of the exponential to obtain

$$a^b a^c = e^{b \operatorname{Log} a} e^{c \operatorname{Log} a} = e^{b \operatorname{Log} a + c \operatorname{Log} a} = e^{(b+c) \operatorname{Log} a} = a^{b+c},$$

as claimed.

b) Now suppose that a branch of log is chosen so that $\log(ab) = \log(a) + \log(b)$, for some fixed $a, b \in \mathbb{C}$. Then we can write

$$(ab)^c = e^{c \log(ab)} = e^{c(\log a + \log b)} = e^{c \log a + c \log b} = e^{c \log a} e^{c \log b} = a^c b^c.$$

Problem 4 (§1.3, 27). Let n be a positive integer, and let $w = e^{2\pi i/n}$. We claim that the n^{th} roots of unity are exactly $\{1, \ldots, w^{n-1}\}$. Let $U = \{z \in \mathbb{C} : z^n = 1\}$ be the set of n^{th} roots of unity and let $V = \{1, \ldots, w^{n-1}\}$. Note that $w^n = (e^{2\pi i/n})^n = e^{2\pi i} = 1$, so w is a root of unity. Then for any $w^k \in V$, we have $(w^k)^n = w^{kn} = (w^n)^k = 1$, so every element of V is also in U, and $V \subseteq U$. Conversely, suppose that $z \in U$. Write $z = re^{i\theta}$, and then by De Moivre's formula, we have $z^n = r^n e^{in\theta} = 1$. $r^n = 1$ implies that $r = 1$, and $e^{in\theta} = 1$ means that $in\theta = 2\pi mi$ for some integer m, by Proposition 1.3.2.vii. so $\theta = 2\pi m/n$ for some integer m. But we can write $m = kn + q$, where k is an integer and $0 \leq q \leq n-1$. Then

$$z = 1 \cdot e^{2\pi mi/n} = e^{2\pi(kn+q)i/n} = e^{2\pi kni/n} e^{2\pi qi/n} = 1 \cdot w^q,$$

so $z = w^q$ for some $0 \leq q \leq n-1$; thus $z \in V$, and $U \subseteq V$. We conclude that $U = V$.

For those who have seen some algebra, this problem demonstrates that w is a generator of the group of n^{th} roots of unity. This group is isomorphic to $\mathbb{Z}/n\mathbb{Z}$ under the map $\phi : \Omega_n \to \mathbb{Z}/n\mathbb{Z}$ defined by $\phi(w^k) = \bar{k}$, the equivalence class of k modulo n.

Problem 5 (§1.4, 2). a) Pick $z_1, z_2 \in \mathbb{C}$, such that $z_j = x_j + iy_j$ for $x_j, y_j \in \mathbb{R}$.
By definition,

$$|\operatorname{Re} z_1 - \operatorname{Re} z_2| = |x_1 - x_2|,$$

while

$$|z_1 - z_2| = |(x_1 - x_2) + i(y_1 - y_2)| = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$$

We can see that $|x_1 - x_2|$ and $|z_1 - z_2|$ are real numbers for which

$$|z_1 - z_2|^2 - |x_1 - x_2|^2 = \left(\sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}\right)^2 - (|x_1 - x_2|)^2 = (y_1 - y_2)^2 \geq 0,$$

which implies that

$$|\operatorname{Re} z_1 - \operatorname{Re} z_2| \leq |z_1 - z_2|.$$

In addition, we know by the triangle inequality that

$$|z_1 - z_2| = |(x_1 + iy_1) - (x_2 + iy_2)| = |(x_1 - x_2) + i(y_1 - y_2)|$$

$$\leq |x_1 - x_2| + |i(y_1 - y_2)| = |x_1 - x_2| + |i||y_1 - y_2| = |\operatorname{Re} z_1 - \operatorname{Re} z_2| + |\operatorname{Im} z_1 + \operatorname{Im} z_2|.$$

b) Suppose that $\lim_{x \to x_0, y \to y_0} u(x,y) = a$ and $\lim_{x \to x_0, y \to y_0} v(x,y) = b$. By definition, given $\epsilon > 0$, there exist $\delta_{ux}, \delta_{uy}, \delta_{vx}, \delta_{vy} > 0$ for which $|x - x_0| < \delta_{ux}, |y - y_0| < \delta_{uy} \Rightarrow |u(x,y) - a| < \epsilon/2$, and $|x - x_0| < \delta_{vx}, |y - y_0| < \delta_{vy} \Rightarrow |v(x,y) - b| < \epsilon/2$. If we let $z := x + iy, z_0 := x_0 + iy_0$, and $\delta := \min\{\delta_{xu}, \delta_{xv}, \delta_{yu}, \delta_{yv}\}$, then by the first half of part a),

$$|z - z_0| < \delta \Rightarrow |x - x_0| = |\operatorname{Re} z - \operatorname{Re} z_0| \leq |z - z_0| < \delta$$

and

$$|y - y_0| = |\operatorname{Re} iz - \operatorname{Re} iz_0| \leq |iz - iz_0| = |i||z - z_0| = |z - z_0| < \delta.$$

This implies in turn that $|u(x,y) - a| < \epsilon/2$ and $|v(x,y) - b| < \epsilon/2$, which means that we can invoke the second half of part a) to conclude that

$$|f(z) - (a+bi)| \leq |\operatorname{Re} f(z) - \operatorname{Re} f(z_0)| + |\operatorname{Im} f(z) - \operatorname{Im} f(z)| = |u(x,y) - u(x_0,y_0)| + |v(x,y) - v(x_0,y_0)| < \epsilon/2 + \epsilon/2 = \epsilon.$$

Thus, $|z - z_0| < \delta \Rightarrow |f(z) - (a+bi)| < \epsilon$, which means that $\lim_{z \to z_0} f(z) = a + bi$.

Now, suppose conversely that $\lim_{z\to z_0} f(z) = a+bi$ for some $a, b \in \mathbb{R}$, such that for every $\epsilon > 0$, there exists $\delta > 0$ for which $|z - z_0| < \delta \Rightarrow |f(z) - (a+bi)| < \epsilon$. If we suppose that $|x - x_0| < \delta/2$ and $|y - y_0| < \delta/2$, then by the second half of part a),

$$|z - z_0| \leq |x - x_0| + |y - y_0| < \delta/2 + \delta/2 = \delta.$$

Therefore, we know by the first half of part a) that

$$|u(x,y) - a| = |\operatorname{Re} f(z) - \operatorname{Re}(a+bi)| \leq |f(z) - (a+bi)| < \epsilon$$

and

$$|v(x,y) - b| = |\operatorname{Re} if(z) - \operatorname{Re} i(a+bi)| \leq |-i||f(z) - (a+bi)| < \epsilon.$$

We conclude that $\lim_{x\to x_0, y\to y_0} u(x,y) = a$, and $\lim_{x\to x_0, y\to y_0} v(x,y) = b$.
If $f(z)$ is continuous then $\lim_{z\to z_0} f(z) = f(z_0)$ for all $z_0 \in \mathbb{C}$. This implies that

$$\lim_{x\to x_0, y\to y_0} u(x,y) = \operatorname{Re} \lim_{z\to z_0} f(z) = \operatorname{Re} f(z_0) = u(x_0, y_0)$$

and

$$\lim_{x\to x_0, y\to y_0} v(x,y) = \operatorname{Im} \lim_{z\to z_0} f(z) = \operatorname{Im} f(z_0) = v(x_0, y_0)$$

for all $x_0, y_0 \in \mathbb{R}$, which means that $u(x,y)$ and $v(x,y)$ are continuous as well. Conversely, if $u(x,y)$ and $v(x,y)$ are continuous, then $\lim_{x\to x_0, y\to y_0} u(x,y) = u(x_0, y_0)$ and $\lim_{x\to x_0, y\to y_0} v(x,y) = v(x_0, y_0)$ for all $x_0, y_0 \in \mathbb{R}$, which implies that

$$\lim_{z\to z_0} f(z) = \lim_{x\to x_0, y\to y_0} u(x,y) + iv(x,y) = u(x_0, y_0) + iv(x_0, y_0) = f(z_0)$$

for all $z_0 \in \mathbb{C}$. Thus, $f(z)$ is continuous if and only if $u(x,y)$ and $v(x,y)$ are both continuous.

Problem 6 (§1.4, 3). By the continuity of f, there exists $\delta > 0$ for which $|z - z_0| < \delta \Rightarrow |f(z) - f(z_0)| < |f(z_0)|$. If $f(z) = 0$, then we can see directly that $|f(z) - f(z_0)| = |0 - f(z_0)| = |f(z_0)|$, which implies that $z \notin D(z_0, \delta$. Thus, f is nonzero on the entire open disk $D(z_0, \delta)$.

Problem 7 (§1.4, 11). Since every nonzero complex number has finite, nonzero absolute value, we can see that if $\lim_{n\to\infty} |z_n| = 0$, then $\lim_{n\to\infty} z_n = 0$. Similarly, if $\lim_{n\to\infty} |z_n| = \infty$, then $\lim_{n\to\infty}$ does not exist. If $|z| > 1$, then by proposition 1.4.4ii,

$$\lim_{n\to\infty} |nz^n| = \lim_{n\to\infty} n \cdot \lim_{n\to\infty} |z|^n = \infty \cdot \infty = \infty.$$

Similarly, if $|z| = 1$, then

$$\lim_{n\to\infty} |nz^n| = \lim_{n\to\infty} n \cdot \lim_{n\to\infty} 1^n = \infty \cdot 1 = \infty.$$

We can apply L'Hopital's rule from ordinary real analysis to treat the case $|z| < 1$, imagining n to be a continuous real variable, to see that one of these two scenarios always occurs:

$$\lim_{n\to\infty} |nz^n| = \lim_{n\to\infty} \frac{n}{|z|^{-n}} = \lim_{n\to\infty} \frac{\frac{\partial}{\partial n}(n)}{\frac{\partial}{\partial n}\exp(-n\log|z|)} = \lim_{n\to\infty} \frac{1}{-n\log|z| \cdot \exp(-n\log|z|) \cdot (-\log|z|)}$$

$$= \lim_{n\to\infty} \frac{1}{n \log^2 |z| \exp(-n \log |z|)} = \lim_{n\to\infty} \frac{|z|^n}{n \log^2 |z|} = 0$$

Therefore, the sequence z_n converges to 0 if $|z| < 1$, and diverges otherwise.