Problem 1 (§4.2, 1). a) Let γ_1 be a circle of radius 2 about the origin. Note that $1/(z+1)^3$ has exactly one singularity, at $z = -1$. The Laurent expansion of this function at -1 has only one term; we have $b_3 = 0$ and all other coefficients 0. In particular $\mathrm{Res}(1/(z+1)^3; -1) = b_1 = 0$. Then

$$\int_{\gamma_1} \frac{dz}{(1+z)^3} = 2\pi i \,\mathrm{Res}(1/(z+1)^3; -1) = 0.$$

§4.2, 2The square γ_2 with vertices $0, 1, 1+i, i$ doesn't even have -1 inside it, so $1/(z+1)^3$ is analytic there. It follows immediately from Cauchy's theorem that

$$\int_{\gamma_2} \frac{dz}{(z+1)^3} = 0.$$

Problem 2 (§4.2, 5). Suppose that $f(z)$ is analytic on and inside a path γ, as in the conditions for the Cauchy integral formula. The function $f(z)/(z-z_0)$ has a single pole at z_0, and by the residue theorem,

$$\int_\gamma \frac{f(z)}{z-z_0} = 2\pi i \,\mathrm{Res}\left(\frac{f(z)}{z-z_0}; z_0\right).$$

If $f(z_0) \neq 0$, then $\mathrm{Res}(f(z)/(z-z_0); z_0) = f(z_0)$ since at z_0, $f(z)$ is nonzero and $z - z_0$ has a zero of order 1. On the other hand, if $f(z_0) = 0$, then $f(z)/(z-z_0)$ has a removable singularity at z_0, and so $\mathrm{Res}(f(z)/(z-z_0); z_0) = 0$. In the first case, the computation with the residue theorem yields

$$\frac{1}{2\pi i}\int_\gamma \frac{f(z)}{z-z_0} = f(z_0),$$

as required. In the second case,

$$\frac{1}{2\pi i}\int_\gamma \frac{f(z)}{z-z_0} = 0 = f(z_0),$$

so the Cauchy integral formula holds in either case.

Problem 3 (§4.2, 14). Let γ be a circle of radius 8 centered at the origin. Recall that $\tan z$ has singularities whereever $\cos z = 0$, that is, exactly at $\pi n + \pi/2$ for $n \in \mathbb{Z}$. The singularities contained in γ are exactly $-5\pi/2, -3\pi/2, \pi/2, \pi/2, 3\pi/2, 5\pi/2$. Since at these points, the numerator of $\tan z = \sin z / \cos z$ is nonzero, so the residue at a singularity z_0 is

$$\frac{\sin(z_0)}{(\cos')(z_0)} = \frac{\sin(z_0)}{-sin(z_0)} = -1.$$

Then

$$\int_\gamma \tan z\, dz = 2\pi i \sum_{i=1}^{6} \mathrm{Res}(\tan z, z_i) = 2\pi i(6(-1)) = -12\pi i.$$

Problem 4. Assume as in the statement of Proposition 4.2.4 (which is proved by this exercise) that f is analytic along γ and has only finitely many singularities outside γ. Choosing a large radius R_0, we can assume that all the singularities are inside a circle Γ of radius R_0 (traversed counterclockwise). Consider the curve σ illustrated on page 263 of Marsden.

By Proposition 4.2.3, we have

$$\int_\Gamma f(z)\,dz = -2\pi i \operatorname{Res}(f;\infty).$$

But then

$$int_\sigma f(z)\,dz = \int_\Gamma f(z)\,dz - \int_\gamma f(z)\,dz = 2\pi i \sum_{k=1}^{n} \operatorname{Res}(f;z_k),$$

where the z_k are the singularities of f outside of γ but inside Γ (these are in fact all the singularities not inside γ, by choice of R_0). Substituting in the computation involving the residue at infinity, we conclude as desired that

$$\int_\gamma f(z)\,dz = -2\pi i \sum_{k=1}^{n+1} \operatorname{Res}(f;z_k),$$

where the z_k $(1 \le k \le n)$ as the singularities outside γ as before, and $z_{k+1} = \infty$.

Problem 5 (§4.3, 5). Note that $\cos(mx)/(1+x^4)$ is even and so

$$\int_0^\infty \frac{\cos mx}{1+x^4}\,dx = \frac{1}{2}\int_{-\infty}^{\infty} \frac{\cos mx}{1+x^4}\,dx = \frac{1}{2}\int_{-\infty}^{\infty} \operatorname{Re}\left(\frac{e^{imz}}{1+z^4}\right)\,dz = \frac{1}{2}\operatorname{Re}\left(\int_{-\infty}^{\infty} \frac{e^{imz}}{1+z^4}\,dz\right).$$

Let $f(z) = e^{imz}/(1+z^4)$. Write $z = x+iy$, with $|z| = R$. Note that if $R > \sqrt[4]{2}$ we have $(R^4-1)/R^4 > 1/2$. Then

$$|f(z)| = \left|\frac{e^{imz}}{1+z^4}\right| = \frac{\left|e^{(im)(x+iy)}\right|}{|1+z^4|} \le \frac{e^{-my}}{R^4-1} < \frac{e^{-my}}{R^4-1}\frac{2(R^4-1)}{R^4} = \frac{2e^{-my}}{R^4}.$$

If $my > 0$, then $e^{-my} < 1$. So if $m > 0$ in the upper half plane we have

$$|f(z)| < \frac{2e^{-my}}{R^4} \le \frac{2}{R^4}.$$

Thus the conditions of Proposition 4.3.6 are satisfied, and

$$\int_{-\infty}^{\infty} f(z)\,dz = 2\pi i \sum \{\text{residues of } f \text{ in the upper half plane}\}.$$

$1+z^4$ vanishes at exactly two points in the upper half plane, $e^{i\pi/4}$ and $e^{3i\pi/4}$. The derivative of the denominator is nonzero at these points, so we compute

$$\operatorname{Res}(f, e^{i\pi/4}) = \frac{e^{im(e^{i\pi/4})}}{4(e^{i\pi/4})} = \frac{e^{im(\sqrt{2}/2+i\sqrt{2}/2)}}{4(-\sqrt{2}/2+i\sqrt{2}/2)} = \frac{e^{-m\sqrt{2}/2}\left(\cos\left(-\frac{m\sqrt{2}}{2}\right) + i\sin\left(\frac{m\sqrt{2}}{2}\right)\right)}{-2\sqrt{2}+2\sqrt{2}i},$$

$$\operatorname{Res}(f, e^{3i\pi/4}) = \frac{e^{im(e^{3i\pi/4})}}{4(e^{3i\pi/4})} = \frac{e^{im(-\sqrt{2}/2+i\sqrt{2}/2)}}{4(\sqrt{2}/2+i\sqrt{2}/2)} = \frac{e^{-m\sqrt{2}/2}\left(\cos\left(-\frac{m\sqrt{2}}{2}\right) + i\sin\left(-\frac{m\sqrt{2}}{2}\right)\right)}{2\sqrt{2}+2\sqrt{2}i}.$$

Thus

$$\int_{-\infty}^{\infty} \frac{e^{imz}}{1+z^4}\,dz = 2\pi i \frac{e^{-m\sqrt{2}/2}}{2\sqrt{2}} \left(\frac{\cos\left(\frac{m\sqrt{2}}{2}\right) + i\sin\left(\frac{m\sqrt{2}}{2}\right)}{-1+i} + \frac{\cos\left(\frac{m\sqrt{2}}{2}\right) - i\sin\left(\frac{m\sqrt{2}}{2}\right)}{1+i} \right)$$

$$= \frac{\pi}{2\sqrt{2}} e^{-m\sqrt{2}/2} \left[(1-i)\left(\cos\left(\frac{m\sqrt{2}}{2}\right) + i\sin\left(\frac{m\sqrt{2}}{2}\right)\right) + \right.$$

$$\left. + (1+i)\left(\cos\left(\frac{m\sqrt{2}}{2}\right) - i\sin\left(\frac{m\sqrt{2}}{2}\right)\right)\right]$$

Then

$$\int_0^{\infty} \frac{\cos(mx)}{1+x^4} = \frac{1}{2}\left(\mathrm{Re}\int_{-\infty}^{\infty} \frac{e^{imz}}{1+z^4}\,dz\right)$$

$$= \frac{\pi}{2\cdot 2\sqrt{2}} e^{-m\sqrt{2}/2} \left(\cos\left(\frac{m\sqrt{2}}{2}\right) + \sin\left(\frac{m\sqrt{2}}{2}\right) + \sin\left(\frac{m\sqrt{2}}{2}\right) + \cos\left(\frac{m\sqrt{2}}{2}\right)\right)$$

$$= \frac{\pi}{2\sqrt{2}} e^{-m\sqrt{2}/2} \left(\cos\left(\frac{m\sqrt{2}}{2}\right) + \sin\left(\frac{m\sqrt{2}}{2}\right)\right).$$

This computation holds for $m \geq 0$. But for $m < 0$, this gives exactly the same integral after replacing m with $-m$, since $\cos(-mx) = \cos(mx)$. Combining the results for these two cases, we obtain

$$\int_0^{\infty} \frac{\cos mx}{1+x^4}\,dx = \frac{\pi}{2\sqrt{2}} e^{-|m|\sqrt{2}/2} \left(\cos\left(\frac{|m|\sqrt{2}}{2}\right) + \sin\left(\frac{|m|\sqrt{2}}{2}\right)\right).$$

Problem 6 (§4.3, 6). (This problem was removed from the set because it requires the Mellin transform, which was not covered in lecture.)

Problem 7 (§4.3, 10). First, note that

$$\int_0^{\pi} \sin^{2n}\theta\,d\theta = \frac{1}{2}\int_{-\pi}^{\pi} \sin^{2n}\theta\,d\theta = \frac{1}{2}\int_0^{2\pi} \sin^{2n}\theta\,d\theta,$$

since $\sin^{2n}\theta$ is odd and has period 2π. Make the standard substitution $z = e^{i\theta}$. Then $\sin\theta = (z - z^{-1})/(2i)$ and $dz = ie^{i\theta}\,d\theta = iz\,d\theta$. First we expand this using the binomial theorem to get the Laurent expansion centered at 0:

$$\int_0^{\pi} \sin^{2n}\theta\,d\theta = \frac{1}{2}\int_0^{2\pi} \sin^{2n}\theta\,d\theta = \frac{1}{2}\int_\gamma \left(\frac{z-z^{-1}}{2i}\right)^{2n} \frac{dz}{iz} = \frac{1}{2i}\frac{1}{2^{2n}(-1)^n}\int_\gamma \sum_{j=0}^{2n} \binom{2n}{j} z^{2n-j}(-z^{-1})^j \frac{dz}{z}$$

$$= \frac{1}{2i}\frac{1}{2^{2n}(-1)^n}\int_\gamma \sum_{j=0}^{2n}\binom{2n}{j}(-1)^j z^{2n-2j-1}\,dz.$$

This function has a single singularity at $z = 0$. The residue there is exactly the coefficient on the z^{-1} term in the preceding expansion, which arises when $n = j$; no other values of j contribute to the residue. This coefficient is exactly

$$a_{-1} = \binom{2n}{n}(-1)^n.$$

Thus

$$\begin{aligned}
\int_0^\pi \sin^{2n}\theta\, d\theta &= \frac{1}{2i}\frac{1}{2^{2n}(-1)^n}\int_\gamma \sum_{j=0}^{2n}\binom{2n}{j}(-1)^j z^{2n-2j-1}\, dz \\
&= \frac{1}{2i}\frac{1}{2^{2n}(-1)^n}\cdot 2\pi i \operatorname{Res}\left(\sum_{j=0}^{2n}\binom{2n}{j}(-1)^j z^{2n-2j-1}; 0\right) \\
&= \frac{1}{2i}\frac{1}{2^{2n}(-1)^n}2\pi i\binom{2n}{n}(-1)^n = \frac{\pi}{2^{2n}}\binom{2n}{n} = \frac{\pi}{2^{2n}}\frac{(2n)!}{n!n!} = \frac{\pi(2n)!}{(2^n n!)^2}.
\end{aligned}$$

Problem 8 (§4.3, 12). (This problem also requires the Mellin transform, so we didn't deduct too many points for mistakes. You should be able to do it if you read the chapter.)

Let $0 < b < 1$. We wish to evaluate

$$\int_0^\infty \frac{1}{x^b(x+1)}\, dx$$

Set $a = 1 - b$. Then $a > 0$ is not an integer and we have

$$\frac{1}{x^b(x+1)} = \frac{x^{-b}}{x+1} = x^{a-1}\frac{1}{x+1}.$$

The function $1/(x+1)$ is analytic except at -1, which is not on the real axis. We can write $f(z) = 1/(z+1) = P(z)/Q(z)$ where $P(z) = 1$ and $Q(z) = z + 1$. Since $0 < a < 1 = \deg Q - \deg P$ and $Q(0) \neq 0$, Corollary 4.3.17 implies that the hypothesis of Proposition 4.3.16 are satisfied. Then

$$\begin{aligned}
\int_0^\infty \frac{dx}{x^b(x+1)} &= \int_0^\infty x^{a-1}\frac{1}{x+1}\, dx = -\frac{\pi e^{-piai}}{\sin(\pi a)}\sum\{\text{residues of } z^{a-1}f(z)\} \\
&= -\frac{\pi e^{\pi i b}e^{-\pi i}}{\sin(\pi - \pi b)}\left((-1)^{a-1}\right) = -\frac{-\pi(-1)^b(-1)}{\sin(\pi b)}(-1)^{-b} = \frac{\pi}{\sin(\pi b)},
\end{aligned}$$

as claimed.