Solutions for Problem Set #5
due October 17, 2003
Dustin Cartwright and Dylan Thurston

1 (B&N 6.5) *Suppose an analytic function* f *agrees with* $\tan x$, $0 \leq x \leq 1$. *Show that* $f(z) = i$ *has no solution. Could* f *be entire? (not graded)*

Since $f(z)$ agrees with $\tan z$ on the points $z_i = 1/i$ which converge to 0, then $f(z)$ and $\tan z$ agree on the domain $\mathbb{C} - \{(n + 1/2)\pi \mid n \in \mathbb{Z}\}$.

In particular, $f(z) = i$ if and only if $\tan z = i$. Let $x = e^{iz}$, and this becomes:

$$\begin{aligned} \frac{x - 1/x}{i(x + 1/x)} &= i \\ \frac{x^2 - 1}{ix^2 + i} &= i \\ x^2 - 1 &= -x^2 - 1 \\ 2x^2 &= 0 \\ x &= 0 \end{aligned}$$

which is impossible, because an exponential is always non-zero.

We know that $f(z)$ cannot be entire for a similar reason. Like $\tan z$ it must approach ∞ as z approaches $\pi/2$, so by continuity, it must have a singularity there.

2 (B&N 6.8) *Show directly that the maximum and minimum moduli of* e^z *are always assumed on the boundary of a compact domain. (not graded)*

Since,

$$|e^{a+bi}| = |e^a||e^{bi}| = e^a$$

for real a, b, and exponentiation as a real function is strictly increasing, then the maximum and minimum of e^z are achieved where the real component is at a maximum or minimum respectively. Therefore for any neighborhood of any z contains a value of e^z with a larger modulus. Thus, the maximum on any compact set D cannot occur in the interior, i.e. it must occur on the boundary.

3 (B&N 6.10) *Suppose f and g are both analytic in a compact domain D. Show that $|f(z)| + |g(z)|$ takes its maximum on the boundary. (10 points)*

Suppose $|f(z)| + |g(z)|$ takes its maximum at z_0. Find real numbers $R_1 > 0, R_2 > 0, \alpha, \beta$ such that:

$$\begin{aligned} R_1 e^{\alpha i} &= f(z_0) \\ R_2 e^{\beta i} &= g(z_0) \end{aligned}$$

Now consider the function $h(z) = e^{-\alpha i} f(z) + e^{-\beta i} g(z)$, which is analytic, and furthermore, I claim that it achieves its maximum at z_0. First note that

$$|h(z)| \leq |e^{-\alpha i}||f(z)| + |e^{-\beta i}||g(z)| = |f(z)| + |g(z)|$$

Second, at z_0, this becomes an equality, because,

$$|h(z_0)| = |e^{-\alpha i} R_1 e^{\alpha i} + e^{-\beta i} R_2 e^{\beta i}| = |R_1 + R_2| = |f(z_0)| + |g(z_0)|$$

and thus, h achieves its maximum at z_0, and that maximum is equal to the maximum of $|f(z)| + |g(z)|$. By the maximum modulus principle, it must also achieve its maximum along the boundary. But by the inequality above, $|f(z)| + |g(z)|$ must also achieve its maximum there.

4 (B&N 6.13) *Suppose $P(z) = a_0 + a_1 z + \ldots + a_n z^n$ is bounded by 1 for $|z| \leq 1$. Show that $|P(z)| \leq |z|^n$ for all $z \gg 1$. (10 points)*

By problem 5.6b from the previous problem set, all of the coefficients a_i have magnitude no greater than 1, and in particular, so does a_n. Now consider the function

$$f(z) = \frac{P(z)}{z^n} = \frac{a_0}{z^n} + \frac{a_1}{z^{n-1}} + \ldots + a_n$$

on the compact domain $\{z \in \mathbb{C} \mid 1 \leq |Z| \leq R\}$. By the maximum modulus principle, this achieves its maximum on the boundary. On the $|z| = 1$ circle, we know f to be bounded by 1. On the $|z| = R$ circle, we have the bound:

$$|f(z)| \leq \frac{|a_0|}{|z|^n} + \frac{|a_1|}{|z|^{n-1}} + \ldots + |a_n| \leq \frac{|a_0|}{R^n} + \frac{|a_1|}{R^{n-1}} + \ldots + 1$$

which gets arbitrary close to 1 for large enough R. Therefore, f is bounded by 1 for all $z \geq 1$, so P is bounded by $|z|^n$ in the same region.

5 *In class, we saw that, for a polynomial $p(z)$ and a simple closed curve Γ, the number of zeros of p inside Γ is equal to the winding number of $p(\Gamma)$ around 0.*

(a) *Show that this is also true for a general analytic function $f(z)$ which is analytic on a disk containing Γ. (10 points)*

Approach 1: "Factor out" the zeros of $f(z)$ to get another function $g(z)$ and show that the winding number of $g(\Gamma)$ around 0 is equal to 0.

Suppose that $f(z)$ is analytic on the disk D. The set of zeros of $f(z)$ has no accumulation points, so $f(z)$ has only a finite number of zeros on D. Let $\alpha_1, \ldots, \alpha_n$ be the zeros (with multiplicities) of $f(z)$. Then the function

$$g(z) = \frac{f(z)}{(z-\alpha_1)\cdots(z-\alpha_n)}$$

is analytic away from the α_i, can be extended continuously to the α_i, and so can be extended to an analytic function on D; furthermore, $g(z)$ is non-zero. Then, as in the case of polynomials, we have that the winding number of a product equals the sum of the winding numbers:

$$\begin{aligned} n(f(\Gamma), 0) &= n(\Gamma - \alpha_1, 0) + \cdots + n(\Gamma - \alpha_n, 0) + n(g(\Gamma), 0) \\ &= n(\Gamma, \alpha_1) + \cdots + n(\Gamma, \alpha_n) + n(g(\Gamma), 0) \end{aligned}$$

Each term $n(\Gamma, \alpha_i)$ is 1 if α_i is contained inside the simple curve Γ, and 0 otherwise, so the first terms give the total number of zeros of $f(z)$ contained inside Γ. It remains to show that the last term, $n(g(\Gamma), 0)$, is 0. Since the disk D is simply connected, there is a homotopy from Γ to a trivial loop inside of D. (See the handout for an explicit homotopy.) Composing with g, we also get a homotopy from $g(\Gamma)$ to a trivial loop; since g is never 0, this is a homotopy inside of $\mathbb{C} \setminus \{0\}$, so the winding number does not change over the course of the homotopy, and $n(g(\Gamma), 0) = 0$.

Approach 2: Show that

$$\int_{f(\Gamma)} \frac{dz}{z} = \int_{\Gamma} \frac{f'(z)}{f(z)} dz$$

and analyze the behaviour of $f'(z)/f(z)$ near a zero of $f(z)$.

Let the curve Γ be parametrized by $z = \gamma(t)$, with $t \in [0,1]$. Then, by definition of the complex line integral,

$$\int_\Gamma \frac{f'(z)}{f(z)} dz = \int_0^1 \frac{f'(\gamma(t))}{f(\gamma(t))} \gamma'(t)\, dt,$$

while

$$\begin{aligned} \int_{f(\Gamma)} \frac{dz}{z} &= \int_0^1 \frac{\frac{d}{dt} f(\gamma(t))}{f(\gamma(t))} dt \\ &= \int_0^1 \frac{f'(\gamma(t))\gamma'(t)}{f(\gamma(t))} dt \end{aligned}$$

These two are the same, so to find the winding number of $f(\Gamma)$ around 0 (given by $\frac{1}{2\pi i} \int_{f(\Gamma)} dz/z$), it suffices to evaluate the integral $\frac{1}{2\pi i} \int_\Gamma f'(z)/f(z)\, dz$. The easiest way to calculate this integral is to apply the Residue Theorem. The poles of $f'(z)/f(z)$ can only occur when the denominator $f(z)$ has a zero. Suppose the denominator $f(z)$ has a zero of order k at α; then we have a power series expansion

$$\begin{aligned} f(z) &= a_k(z-\alpha)^k + a_{k+1}(z-\alpha)^{k+1} + \cdots \\ f'(z) &= k a_k (z-\alpha)^{k-1} + (k+1) a_{k+1} (z-\alpha)^k + \cdots \\ \frac{f'(z)}{f(z)} &= k(z-\alpha)^{-1} + \cdots \end{aligned}$$

so $f'(z)/f(z)$ has a pole with residue k at α. Now $\frac{1}{2\pi i} \int_\Gamma f'(z)/f(z)\, dz$ is the sum of the residues of the poles of $f'(z)/f(z)$ at the poles included inside Γ (since Γ is a simple closed curve, so all the winding numbers are 1), which, by the computation above, is the sum of the orders of the zeros of $f(z)$ inside of Γ, as desired.

(b) *Define the "inside" of a general curve Γ to be the set of points $z \in \mathbb{C}$ so that $n(\Gamma, z) \neq 0$. Show that, under the assumptions above, every point in the inside of Γ gets mapped to the inside of $f(\Gamma)$. (10 points)*

(Recall that the assumptions above were that f is analytic on a disk and that Γ is a simple closed curve contained inside the disk.)

First let us generalize the result from the first part. We found that the number of zeros of $f(z)$ inside of Γ was equal to the winding number of $f(\Gamma)$

around 0. By replacing $f(z)$ by $f(z) - a$, we can also see that the number of times f takes the value a inside Γ (with multiplicity) equals the winding number of $f(\Gamma)$ around a. In particular, for any point z on the inside of Γ, the value $f(z)$ is obtained at least once, so the winding number of $f(\Gamma)$ around $f(z)$ is at least one, and so $f(z)$ is on the inside of $f(\Gamma)$ as defined in the statement of the problem.

Note that it is crucial that Γ is a simple closed curve: if not, we would have to take into account the winding number of Γ around the different points at which f takes the value a; if Γ has negative winding number around some points, we could get cancellation.

(c) *Show how this result generalizes the Maximum Modulus principle. (10 points)*

Let R be the maximum value of $|z|$ on the curve $f(\Gamma)$. Then every point on the inside of $f(\Gamma)$ is contained inside the disk D_R of radius R around 0. There are several ways to see this. For instance, for a outside of D_R, a straight line from a out to infinity does not intersect $f(\Gamma)$, so the winding number is 0 according to the counting intersections criterion. But by the previous part, every point on the inside of Γ maps to the inside of $f(\Gamma)$, and so in particular maps inside D_R, and so the maximum absolute value of the function f on the region bounded by Γ occurs on Γ itself.

6 (problem 42 from Needham p. 121) *Consider the mapping $z \mapsto w = P_n(z)$, where $P_n(z)$ is a general polynomial of degree $n \geq 2$. Let S_q be the set of points in the z-plane that are mapped to a particular point q in the w-plane. Show that the centroid of S_q is independent of the choice of q, and is therefore a property of the polynomial itself. (5 points extra credit)*

Note that the statement is, in general, not correct, because it doesn't account for the multiplicity of the roots. Therefore, we assume that the points of S_q are weighted by the multiplicity of that point as a zero of $P_n - q$.

Let $P_n = c_n x^n + \ldots + c_0$ be such a polynomial, and then S_q is the same as the set of roots of the polynomial $P_n - q$. Call this polynomial Q. By the Fundamental Theorem of Algebra, we can factor Q as $c(x - a_1) \ldots (x -$

a_n). The centroid of S_q (accounting for multiplicity as above), is $\sum a_i/n$. However, if we look at the two highest terms of Q we find that they are:

$$cx^n + c(-a_1 - \ldots - a_n)x^{n-1}$$

But if $n \geq 2$, then these two terms are also the same as the corresponding terms for P_n:

$$c_n x^n + c_{n-1}x^{n-1}$$

Therefore, the centroid is

$$-\frac{c_{n-1}}{nc_n}$$

which is independent of the choice of q.

7 *Let $f(z)$ be an arbitrary function on $\mathbb{C}$ with values in $\mathbb{C}$ with two continuous (real) derivatives. In other words, near each $z_0 \in \mathbb{C}$, f can be written as*

$$\begin{aligned} f(z_0 + x + iy) &= f(z_0) + f_x(z_0)x + f_y(z_0)y \\ &\quad + f_{xx}(z_0)\frac{x^2}{2} + f_{xy}(z_0)xy + f_{yy}(z_0)\frac{y^2}{2} + \epsilon(x, y) \end{aligned}$$

where $\lim_{x,y\to 0} \epsilon(x, y)/|x + iy|^2 = 0$.

Define the Laplacian *of f as*

$$\delta f(z) = \left(\left(\frac{\partial}{\partial x}\right)^2 + \left(\frac{\partial}{\partial y}\right)^2\right) f(z) = f_{xx}(z) + f_{yy}(z).$$

(a) *Show that*

$$\frac{1}{2\pi}\int_0^{2\pi} f(z_0 + re^{i\theta})d\theta = f(z_0) + \frac{1}{4}r^2\delta f(z_0) + \epsilon'(r)$$

where $\lim_{r\to 0} \epsilon'(r)/r^2 = 0$.

Let $f(z)$ be an arbitrary function on $\mathbb{C}$ with values in $\mathbb{C}$ with two continuous (real) derivatives. In other words, near each $z_0 \in \mathbb{C}$, f can be written as

$$\begin{aligned} f(z_0 + x + iy) &= f(z_0) + f_x(z_0)x + f_y(z_0)y \\ &\quad + f_{xx}(z_0)\frac{x^2}{2} + f_{xy}(z_0)xy + f_{yy}(z_0)\frac{y^2}{2} + \epsilon(x, y) \end{aligned}$$

where $\lim_{x,y\to 0} \epsilon(x,y)/|x+iy|^2 = 0$.

(b) *Show that if f is harmonic, then δf is identically* 0.

If f is harmonic, then the average value on any circle centered at z_0 equals the value at the center:

$$\begin{aligned} f(z_0) &= \frac{1}{2\pi}\int_0^{2\pi} f(z_0 + re^{i\theta})d\theta \\ &= f(z_0) + \frac{r^2}{4}\Delta f(z_0) + \epsilon'(r). \end{aligned}$$

Subtracting $f(z_0)$ from both sides and dividing by r^2, we find:

$$0 = \frac{1}{4}\Delta f(z_0) + \frac{\epsilon'(r)}{r^2}$$

Taking the limit as $r \to 0$, we get the desired result.

(c) *Conversely, use the factorization*

$$\left(\left(\frac{\partial}{\partial x}\right)^2 + \left(\frac{\partial}{\partial y}\right)^2\right) = \left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right)\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right)$$

to show that if $\delta f = 0$, then there exist analytic functions $g(z)$ and $h(z)$ so that

$$f(z) = g(z) + \overline{h}(z)$$

That is, the only harmonic functions are sums of analytic and anti-analytic functions.

Consider the function

$$p(z) = \left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right) f(z).$$

Since f satisfies $\Delta f = 0$, we have

$$\Delta f(z) = \left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right) p(z) = 0.$$

But this says that

$$p_x(z) + ip_y(z) = 0$$

or, in other words, that g satisfies the Cauchy-Riemann equations $p_y = ip_x$. Since the second derivatives of p exist, p_x and p_y are continuous, and so $p(z)$ is complex-differentiable and therefore analytic.

Now suppose that we had

$$f(z) = g(z) + \bar{h}(z)$$

as desired. Then we would have

$$p(z) = \left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right) f(z) = (g_x - ig_y)(z) + (\bar{h}_x - i\bar{h}_y)(z).$$

But if g is analytic, then $g_y(z) = ig_x(z)$ and $g_x(z) = g'(z)$, while if $\bar{h}$ is anti-analytic, $\bar{h}_y(z) = -i\bar{h}_x(z)$, so this becomes

$$p(z) = 2g_x(z) = 2g'(z).$$

To return to what we actually have, this suggests that we define g in terms of p:

$$g(z) = \frac{1}{2}\int_0^z p(w)\,dw.$$

Now $\bar{h}(z)$ is determined: define

$$\bar{h}(z) = f(z) - g(z).$$

Let's check that $\bar{h}$ is anti-analytic:

$$\begin{aligned}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right)\bar{h}(z) &= \left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right) f(z) - \left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right) g(z) \\ &= p(z) - 2g'(z) \\ &= 0.\end{aligned}$$

We used twice the fact, which is easy to check, that a function is anti-analytic if and only if it has continuous first partial derivatives and satisfies the anti-Cauchy-Riemann equations:

$$\bar{h}_y(z) = -i\bar{h}_x(x).$$