Solutions for Problem Set #4

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(B&N 4.3) *Evaluate* $\int_C f$ *where* $f(z) = 1/z$ *as in Example 2, and* C *is given by* $z(t) = \sin t + i \cos t, 0 \leq t \leq 2\pi$. *Why is the result different from that of Example 2? (not graded)*

We can substitute the path given:

$$\begin{aligned}
\int_C f\,dz &= \int_0^{2\pi} \frac{1}{z}\frac{dz}{dt}\,dt \\
&= \int_0^{2\pi} \frac{1}{\sin t + i\cos t}(\cos t - i\sin t)\,dt \\
&= \int_0^{2\pi} \frac{(\cos t - i\sin t)(\sin t - i\cos t)}{(\sin t + i\cos t)(\sin t - i\cos t)}\,dt \\
&= \int_0^{2\pi} \frac{\cos t\sin t - i\sin^2 t - i\cos^2 t - \sin t\cos t}{\sin^2 t + \cos^2 t}\,dt \\
&= \int_0^{2\pi} -i\,dt \\
&= -2\pi i
\end{aligned}$$

This is different because the path makes the circle in the opposite direction, so it has the opposite sign.

(B&N 4.8) *Show that* $\int_C z^k dz = 0$ *for any integer* $k \neq -1$ *and* $C : z = Re^{i\theta}, i \leq \theta \leq 2\pi$.

a. *by showing that* z^k *is the derivative of a function analytic throughout* C. *(10 points)*

Define $F(z) = z^{k+1}/(k+1)$ which is defined an analytic for $\mathbb{C} - \{0\}$. The derivative is z^k, which is equal to f. Therefore a integral of f is defined by the difference between the values of F at the endpoints. Since the endpoints are the same, the difference, and thus the integral are 0.

b. *directly, using the parametrization of* C. *(10 points)*

$$\begin{aligned}
\int_C z^k\,dz &= \int_0^{2\pi} (Re^{i\theta})^k iRe^{i\theta}\,d\theta \\
&= R^{k+1} i \int_0^{2\pi} e^{(k+1)i\theta}\,d\theta \\
&= R^{k+1} i \left[\frac{1}{(k+1)i} e^{(k+1)i\theta}\right]_{\theta=0}^{2\pi} \\
&= 0
\end{aligned}$$

because $k+1$ is a non-zero integer, so $e^{(k+1)i2\pi} = 1 = e^0$.

(B&N 5.3) *f is called an* odd *function if $f(z) = -f(-z)$ for all z; f is called* even *if $f(z) = f(-z)$.*

a. *Show that an odd entire function has only odd terms in its power series about $z = 0$. (not graded)*

Since f is odd,

$$f(z) = \frac{f(z) + f(z)}{2} = \frac{f(z) - f(-z)}{2}$$

If we suppose f has the power series expansion $\sum a_n z^n$, then

$$\begin{aligned}
f(z) &= \frac{\sum a_n z^n - \sum a_n (-1)^n z^n}{2} \\
&= \frac{1}{2} \sum_{n \text{ odd}} 2a_n z^n \\
&= \sum_{n \text{ odd}} a_n z^n
\end{aligned}$$

Therefore, since power series are unique, f's power series has only odd coefficients.

b. *Prove an analogous result for even functions. (not graded)*

Since f is even,

$$f(z) = \frac{f(z) + f(z)}{2} = \frac{f(z) + f(-z)}{2}$$

If we suppose f has the power series expansion $\sum a_n z^n$, then

$$\begin{aligned} f(z) &= \frac{\sum a_n z^n + \sum a_n (-1)^n z^n}{2} \\ &= \frac{1}{2} \sum_{n \text{ even}} 2 a_n z^n \\ &= \sum_{n \text{ even}} a_n z^n \end{aligned}$$

Therefore, since power series are unique, f's power series has only even coefficients.

An alternative way to prove these two statements at once is to use induction. First note that if f is odd, $f(0) = f(-0) = -f(0)$, so $f(0)$ must equal 0. Now, if f is odd, then by the chain rule,

$$f'(z) = \frac{df(z)}{dz} = \frac{d(-f(-z))}{dz} = -f'(-z)(-1) = f'(-z)$$

Therefore f' is an even function. Furthermore, if g is even, then

$$g'(z) = \frac{dg(z)}{dz} = \frac{dg(-z)}{dz} = g'(-z)(-1) = -g'(-z)$$

so g' is odd. Therefore, by induction, $f^{(2k)}(0) = 0$, and $g^{(2k+1)}(0) = 0$, so the corresponding power series coefficients are zero. Thus, f has only odd terms, and g has only even terms.

(B&N 5.6a.) *Suppose an entire function f is bounded by M along $|z| = R$. Show that the coefficients C_k in its power series expansion about 0 satisfy*

$$|C_k| \leq \frac{M}{R^k}$$

(10 points)

By the proof of the existence of a Taylor Series, the kth coefficient is:

$$\frac{1}{2\pi i} \int_C \frac{f(z)}{z^{k+1}}\, dz$$

Now we apply the ML-inequality. The absolute value of the inside of the integral is

$$\frac{|f(z)|}{R^{k+1}}$$

because $|z| = R$ by hypothesis. So, the maximum value of this is $MR^{-(k+1)}$. Since the length of the path is $2\pi R$, the absolute value of the kth coefficient is

$$\leq \left|\frac{1}{2\pi i}\right| 2\pi R \frac{M}{R^{k+1}} = \frac{M}{R^k}$$

b. *Suppose a polynomial is bounded by* 1 *in the unit disc. Show that all its coefficients are bounded by* 1. *(5 points)*

Apply the result from part a, with $R = 1$, $M = 1$.

(B&N 5.7) *An alternate proof of Louiville's Theorem. Suppose that* $|f(z)| \leq A + B|z|^k$ *and that* f *is entire. Show then that all the coefficients* $C_j, j > k$, *ini its power series expansion are* 0. *(10 points)*

Apply the result from problem 5.6a, to find a bound on C_j. With an arbitrary value of R, we can take $M = A + BR^k$, so that:

$$C_j \leq \frac{A + BR^k}{R^j} = \frac{A}{R^j} + \frac{B}{R^{j-k}}$$

Since $j > k \geq 0$, the right side gets arbitrarily close to 0 as R gets large. Therefore, C_j must be 0.

(B&N 5.12) *Show that every real polynomial is equal to a product of real linear and quadratic polynomials. (10 points)*

Let p be an arbitrary real polynomial. By the Fundamental Theorem of Algebra, we can write $p(x) = k(x - \alpha_1) \ldots (x - \alpha_n)$, where the $\{\alpha_i\}$ are the roots of p. By a previous homework problem, if α_i is a root then $\overline{\alpha_i}$ is also a root. There are two possibilities. If α_i is real, then $(x - \alpha_i)$ is a real polynomial factor. If not, then we can pair each non-real root with its conjugates to get factors of the form:

$$(x - \alpha_i)(x - \overline{\alpha_i}) = x^2 - 2\operatorname{Re}(\alpha_i)x + |\alpha_i|^2$$

which is a real quadratic factor. Thus, p can be written as the product of real linear and quadratic terms.

(B&N 4.6) *Show that if* f *is a continuous* real-valued *function and* $f \ll 1$, *then*

$$\int_{|z|=1} f(z)\,dz \ll 4$$

(5 points extra credit)

Note: For anyone else who was confused by the symbol $\ll$, my sources indicate that $a \ll b$ means $|a| \leq |b|$ (at least when Bak and Newman use it).

Write $\int_{|z|=1} f(z)\,dz$ in polar form as $Re^{i\phi}$, with R, ϕ real. If we parametrize the integral in the standard way,

$$\begin{aligned}
\int_0^{2\pi} f(e^{i\theta}) ie^{i\theta}\,d\theta &= Re^{i\phi} \\
i\int_0^{2\pi} f(e^{i\theta}) e^{i(\theta-\phi)}\,d\theta &= R \\
i\int_0^{2\pi} f(e^{i\theta})\cos(\theta-\phi)\,d\theta - \int_0^{2\pi} f(e^{i\theta})\sin(\theta-\phi)\,d\theta &=
\end{aligned}$$

But, since R is real, the first term, which is purely imaginary, must be zero. Thus,

$$\begin{aligned}
|R| &\leq \int_0^{2\pi} |f(e^{i\theta})||\sin(\theta-\phi)|\,d\theta \\
&\leq \int_0^{2\pi} |\sin(\theta-\phi)|\,d\theta \\
&= \int_{-\phi}^{2\pi-\phi} |\sin(\theta)|\,d\theta \\
&= \int_{-\phi}^{0} |\sin(\theta)|\,d\theta + \int_0^{2\pi-\phi} |\sin(\theta)|\,d\theta \\
&= \int_{2\pi-\phi}^{2\pi} |\sin(\theta)|\,d\theta + \int_0^{2\pi-\phi} |\sin(\theta)|\,d\theta
\end{aligned}$$

because $\sin(\theta)$ is periodic. Thus,

$$\begin{aligned}
&= \int_0^{2\pi} |\sin(\theta)|\,d\theta \\
&= \int_0^{\pi} \sin(\theta)\,d\theta + \int_\pi^{2\pi} -\sin(\theta)\,d\theta \\
&= 2+2=4
\end{aligned}$$

(B&N 5.10) *Prove that a nonconstant entire function cannot satisfy the two equations:*

i. $f(z+1) = f(z)$

ii. $f(z+i) = f(z)$

for all z. (5 points extra credit)

Consider the region $D = \{a + bi \mid 0 \leq a \leq 1, 0 \leq b \leq 1\}$. This is compact, so its image under f is compact, and thus bounded, by say M. For any other $a+bi \in \mathbb{C}$, we can write $a+bi = (n+a') + (m+b')i$, where n, m are integers, and a', b' are in D. By repeated application of the equations i, ii, $f(a+bi) = f(a'+b'i) \leq M$. In other words, f is bounded, and therefore constant.

(B&N 5.13) *Suppose P is a polynomial such that $P(z)$ is real if and only if z is real. Prove that P is linear. (10 points extra credit)*

Write $P(x+iy) = u(x,y) + iv(x,y)$ as usual. We know that $v(x,y) = 0$ if and only if $y = 0$. By the intermediate value theorem, this means that on the upper half-plane, i.e. when $y > 0$, $v(x,y)$ must be entirely greater than zero, or entirely less than zero. Now consider the partial $v_y(x,0)$. If it is negative for some x-value, then we can find a small positive y such that $v(x,y)$ is negative. Therefore, if $v(x,y)$ is positive in the upper half-plane, then $v_y(x,0)$ is non-negative, and by the same reasoning, if $v(x,y)$ is negative in the upper half-plane, then $v_y(x,0)$ is non-positive.

But the Cauchy-Riemann tells us that $u_x(x,0) = v_y(x,0)$, so $u(x,0)$ must be either a decreasing or an increasing function of x. Thus the polynomial P has at most one root on the real axis. But since zero is a real number, all the zeros of P occur along the real line. Therefore, by the fundamental theorem of algebra, $P(z) = c(z-\alpha)^n$ for some real numbers α, c. But

$$P(\alpha + e^{\pi i/n}) = ce^{\pi i} = -c \in \mathbb{R}$$

and if $n > 1$, $\alpha + e^{\pi i/n}$ is not a real number, so this contradicts our hypothesis. Therefore, $n = 1$, and so P is a linear polynomial.