Math 113 Problem Set 7

October 27, 2003

Due October 31, 2003

1. In class we found that the function $\pi\cot(\pi z)$ has a pole at every integer n with residue 1. One natural approach to constructing such a function is to consider a sum over the integers n of a function with a pole at n: consider

$$f(z) = \sum_{n=-\infty}^{\infty} \frac{1}{z-n} \tag{1}$$

   (a) Show that this approach does not work, by showing that the sum above diverges for all $z \in \mathbb{C}$.

   (Recall that a sum $\sum_{n=-\infty}^{\infty} a_n$ converges if and only if $\lim_{N,M\to\infty} \sum_{n=-N}^{M} a_n$ exists, which in turn is true if and only if there is a value A so that for every $\epsilon \to 0$, there are N_0 and M_0 so that, for $N \geq N_0$ and $M \geq M_0$, $\sum_{n=-N}^{M} a_n$ is within ϵ of the limiting value A.)

   We can be a little more clever, by taking both endpoints of the sum simultaneously to infinity:

$$g(z) = \lim_{N\to\infty} \sum_{n=-N}^{N} \frac{1}{z-n} = \frac{1}{z} + \sum_{n=1}^{N} \left(\frac{1}{z-n} + \frac{1}{z+n} \right) \tag{2}$$

   (b) Show that the sum in (2) converges, and that the result is an analytic function with a pole at every integer with residue 1 which satisfies $g(z+1) = g(z)$.

   (c) Evaluate the sum to show that, in fact, $g(z) = \pi\cot(\pi z)$.

   In summary, although $\pi\cot(\pi z)$ is not the only function with a pole of residue 1 at each integer, it is in some sense the most natural one.

2. Show that $\frac{\pi}{\sin(\pi z)}$ has a pole at every integer, with a residue at n of $(-1)^n$.

3. Bak and Newman, Chapt. 11, p. 147, problem 6.

4. Bak and Newman, Chapt. 11, p. 147, problem 5.

5. Bak and Newman, Chapt. 11, p. 147, problem 1(d).
6. Needham, p. 446, problem 3.
7. Needham, p. 446, problem 1.
8. (Optional) Needham, p. 447, problem 4.