MATH 113 PROBLEM SET 5

Due October 17, 2003

(1) Bak and Newman, Chapter 6, page 79, number 5.
(2) Bak and Newman, Chapter 6, page 79, number 8.
(3) Bak and Newman, Chapter 6, page 79, number 10.
(4) Bak and Newman, Chapter 6, page 79, number 13.
(5) In class, we saw that, for a polynomial $p(z)$ and a simple closed curve Γ, the number of zeros of p inside Γ is equal to the winding number of $p(\Gamma)$ around 0.
   (a) Show that this is also true for a general analytic function $f(z)$ which is analytic on a disk containing Γ. There are at least two approaches:
      - "Factor out" the zeros of $f(z)$ to get another function $g(z)$ and show that the winding number of $g(\Gamma)$ around 0 is equal to 0; or
      - Show that
      $$\int_{f(\Gamma)} \frac{dz}{z} = \int_{\Gamma} \frac{f'(z)\,dz}{f(z)}$$
      and analyze the behaviour of $f'(z)/f(z)$ near a zero of $f(z)$.

   You may use either approach, or optionally both, or any other method you wish. (There is also a topological proof in Needham, Section 7.IV, which looks at what is true for more general maps of the plane to itself, and what is special about the case of analytic functions. That proof is well worth reading if you have a topological bent.)
   (b) Define the "inside" of a general curve Γ to be the set of points $z \in \mathbb{C}$ so that $n(\Gamma, z) \neq 0$. Show that, under the assumptions above, every point in the inside of Γ gets mapped to the inside of $f(\Gamma)$.
   (c) Show how this result generalizes the Maximum Modulus principle.
(6) (Optional) In Needham, Section 1.VIII, there is an elementary proof of the Mean Value Theorem which may give some more insight into why it is true. Read it, and then do Exercise 42 on page 121.
(7) (Optional) Let $f(z)$ be an arbitrary function on $\mathbb{C}$ with values in $\mathbb{C}$ with two continuous (real) derivatives. In other words, near each $z_0 \in \mathbb{C}$, f can be written as

$$f(z + x_0 + iy_0) = f(z_0) + f_x(z_0)x_0 + f_y(z_0)y_0 + f_{xx}(z_0)\frac{x_0^2}{2} + f_{xy}xy + f_{yy}\frac{y_0^2}{2} + \epsilon(x_0, y_0)$$

where $\lim_{x_0,y_0\to 0} \epsilon(x_0, y_0)|x_0 + iy_0|^2 = 0$.

Define the *Laplacian* of f as

$$\Delta f(z) = \left(\left(\frac{\partial}{\partial x}\right)^2 + \left(\frac{\partial}{\partial y}\right)^2\right) f(z) = f_{xx}(z) + f_{yy}(z).$$

   (a) Show that
   $$\frac{1}{2\pi}\int_0^{2\pi} f(z_0 + re^{i\theta})d\theta = f(z_0) + \frac{1}{4}r^2\Delta f(z_0) + \epsilon'(r)$$

where $\lim_{r \to 0} \epsilon' * r^2 = 0$.

(b) Show that if f is harmonic, then Δf is identically 0.

(c) Conversely, use the factorization

$$\left(\frac{\partial}{\partial x}\right)^2 + \left(\frac{\partial}{\partial y}\right)^2 = \left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right)\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right)$$

to show that if $\Delta f = 0$, then there exist analytic functions $g(z)$ and $h(z)$ so that

$$f(z) = g(z) + i\bar{h}(z).$$

That is, the only harmonic functions are sums of analytic and anti-analytic functions. (Hint: Start by showing that $g(z) = \left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right) f(z)$ is analytic.)