

MATH 113 MIDTERM EXAM: SOLUTIONS

(1) Consider the map $f(z) = 1/z$.

- (a) For which points $z \in \mathbb{C}$ does f shrink a sufficiently small neighborhood of z ? That is, for which points z is

$$\lim_{\epsilon \rightarrow 0} \frac{\text{Area}(f(D_\epsilon(z)))}{\text{Area}(D_\epsilon(z))} < 1?$$

You need not prove your answer.

The map $f(z)$ on the complex plane near the point z is infinitesimally expanded and twisted by $f'(z)$. In particular, a small disk is shrunk exactly when $|f'(z)| < 1$. Since $f'(z) = -\frac{1}{z^2}$, this happens for $|z| > 1$.

- (b) If $z = x + iy$, what are the images of the lines $x = \text{const}$ and the lines $y = \text{const}$? Draw the images and identify the geometric objects they trace out, with proof.

The point $z = a + iy$ on the line $x = a$ is mapped to the point

$$\frac{1}{z} = \frac{a - iy}{a^2 + y^2},$$

with real and imaginary parts

$$u = \frac{a}{a^2 + y^2}, \quad v = -\frac{y}{a^2 + y^2}.$$

These satisfy

$$u^2 + v^2 = \frac{a^2 + y^2}{(a^2 + y^2)^2} = \frac{1}{a^2 + y^2} = \frac{u}{a}$$

or, equivalently,

$$\left(u - \frac{1}{2a}\right)^2 + v^2 = \frac{1}{4a^2}$$

which is the equation of a circle with radius $1/2|a|$ and center $(1/2a, 0)$; that is, the image of a vertical line is a circle with center on the x -axis, passing through the origin. Similarly, the image of the horizontal line $y = b$ is a circle of radius $1/2|b|$ with center at $(0, -1/2b)$.

- (c) What is the image of the first quadrant? Where does the boundary of the first quadrant go (with orientation)?

(2) Evaluate the following sums and integrals:

(b) $\int_{-\infty}^{\infty} \frac{dx}{x^4 + 1}$

By the arguments in Bak and Newman, pages 130–132, this is equal to the integral around a closed curve that runs along the real axis and returns around a large loop in the upper half plane, which in turn is equal to a sum over the residues of the poles included:

$$\begin{aligned} \int_{-\infty}^{\infty} \frac{dz}{z^4 + 1} &= 2\pi i \left(\operatorname{Res}_{z=e^{\pi i/4}} \frac{1}{z^4 + 1} + \operatorname{Res}_{z=e^{3\pi i/4}} \frac{1}{z^4 + 1} \right) \\ &= 2\pi i \left(\frac{1}{4z^3} \Big|_{z=e^{\pi i/4}} + \frac{1}{4z^3} \Big|_{z=e^{3\pi i/4}} \right) \\ &= \frac{\sqrt{2}\pi}{2}. \end{aligned}$$

(c) $\int_{-\infty}^{\infty} \frac{\cos ax}{x^2 + b^2} dx$ (with $a, b \in \mathbb{R}$)

First assume $a > 0$, $b > 0$ for simplicity. As in class, this is the real part of an integral involving e^{iax} :

$$\int_{-\infty}^{\infty} \frac{\cos ax}{x^2 + b^2} dx = \operatorname{Re} \left(\int_{-\infty}^{\infty} \frac{e^{iax}}{x^2 + b^2} dx \right)$$

Since $a > 0$, if we close off the contour with a large half-circle in the upper half-plane, the integral doesn't change, so the value of the integral equals the sum over the residues at the poles in the upper half-plane. Since the only pole is at $x = i|b|$, we have

$$\begin{aligned} \int_{-\infty}^{\infty} \frac{\cos ax}{x^2 + b^2} dx &= \operatorname{Re} \left(2\pi i \operatorname{Res}_{x=ib} \frac{e^{iax}}{(x+ib)(x-ib)} \right) \\ &= \operatorname{Re} \left(2\pi i \frac{e^{-ab}}{2ib} \right) \\ &= \frac{\pi e^{-ab}}{b}. \end{aligned}$$

In general, it's easy to see that the answer doesn't depend on the sign of a or b :

$$\int_{-\infty}^{\infty} \frac{\cos ax}{x^2 + b^2} dx = \frac{\pi e^{-|ab|}}{|b|}.$$

(d) $\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)^3}$ (Hint: the function $\frac{\pi}{\sin \pi z}$ is useful for summing alternating series).

Note that the summand is symmetric with respect to $n \mapsto -1 - n$, so the sum we are looking for is

$$S = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)^3} = \frac{1}{2} \sum_{n=-\infty}^{\infty} \frac{(-1)^n}{(2n+1)^3}.$$

Next, observe that the function

$$f(z) = \frac{\pi}{(2z+1)^3 \sin \pi z}$$

has a pole at every integer n with residue $(-1)^n/(2n+1)^3$, in addition to a pole at $-1/2$. Thus

$$\frac{1}{2\pi i} \oint_{C_N} f(z) dz = \sum_{n=-N}^N \frac{(-1)^n}{(2n+1)^3} + \operatorname{Res}_{z=-1/2} f(z)$$

where C_N is a circle of radius $N + 1/2$ around the origin. As $N \rightarrow \infty$, the integral goes to 0 (as proved in Bak and Newman, page 142), so in the limit we have

$$0 = 2S + \operatorname{Res}_{z=-1/2} f(z)$$

or

$$S = -\frac{1}{2} \operatorname{Res}_{z=-1/2} f(z).$$

To evaluate the residue, it is most convenient to translate the function so we can do a power series expansion around 0, by substituting $w = z - 1/2$:

$$\operatorname{Res}_{z=-1/2} \frac{\pi}{(2z+1)^3 \sin \pi z} = \operatorname{Res}_{w=0} \frac{\pi}{(2w)^3 (-\cos \pi w)}$$

We need the first few terms of the Laurent series

$$\begin{aligned} \cos z &= 1 - \frac{z^2}{2} + \frac{z^4}{24} - \dots \\ \frac{1}{\cos z} &= 1 + \frac{z^2}{2} + \frac{5z^4}{24} + \dots \\ \frac{\pi}{\cos \pi z} &= \pi + \frac{\pi^3 z^2}{2} + \frac{5\pi^5 z^4}{24} + \dots \\ -\frac{\pi}{(2w)^3 \cos \pi w} &= \frac{\pi}{8w^3} + \frac{\pi^3}{16w} + \frac{5\pi^5 w}{192} + \dots \end{aligned}$$

(Note that we carried out the series to one more term than necessary. The easiest way to go from the series for $\cos z$ to $1/\cos z$ is to solve for the coefficients term-by-term.) Therefore, we have

$$\begin{aligned} S &= -\frac{1}{2} \operatorname{Res}_{z=-1/2} \frac{\pi}{(2z+1)^3 \sin \pi z} \\ &= \frac{1}{2} \operatorname{Res}_{w=0} \frac{\pi}{(2w)^3 \cos \pi w} \\ &= \frac{\pi^3}{32}. \end{aligned}$$

- (3) Find the doubly-infinite expansion for the function $f(z) = \frac{1}{(2z-1)(z-2)}$ which converges inside the annulus $\frac{1}{2} < |z| < 2$.

We can expand out $f(z)$ using a partial fraction expansion:

$$\begin{aligned} f(z) &= \frac{1}{3(z-2)} - \frac{2}{3(2z-1)} \\ &= \sum_{n=0}^{\infty} -\frac{1}{3 \cdot 2^{n+1}} z^n + \sum_{n=1}^{\infty} -\frac{1}{3 \cdot 2^{n-1}} z^{-n} \\ &= \sum_{n=-\infty}^{\infty} -\frac{1}{3 \cdot 2^{|n+1|}} z^n. \end{aligned}$$

Alternatively, you can guess the expansion above, check that it converges in the right range, and multiply by $(z-2)(2z-1)$ and see that you get 1. Finally, you can take the expansions for $2z-1$ and $z-2$ and multiply them together. Since both series converge absolutely, you can rearrange terms in the product to find the same expansion.

- (4) Find $\oint \bar{z} dz$ around
 (a) The boundary of an arbitrary rectangle and
 (b) The boundary of an arbitrary circle.

Does the result have a geometric significance? (Hint: you might want to show that the integral is invariant under translation of the contour first.)

Most students parametrized the curves in order to do this; this was the intended solution, but there's a more general solution:

$$\begin{aligned}\oint \bar{z} dz &= \oint (x - iy)(dx + idy) \\ &= \left(\oint x dx + y dy \right) + i \left(\oint x dy - y dx \right)\end{aligned}$$

The real part of this integral vanishes since it is a total derivative of $x^2 + y^2$. For the imaginary part, Stoke's theorem tells us that for a curve C enclosing a region R ,

$$\oint_C x dy = \iint_R dx dy = \text{Area}(R).$$

Similarly

$$\oint_C y dx = \iint_R dy dx = -\text{Area}(R).$$

Thus

$$\oint_C \bar{z} dz = 2i \text{Area}(R).$$

(5) Suppose that $f(z)$ is a meromorphic function on all of \mathbb{C} , and that $\lim_{z \rightarrow \infty} f(z) = \infty$. (Specifically, this means that for every $r > 0$, there is an $R > 0$ so that for $|z| > R$, $|f(z)| > r$.)

(a) Show that $f(z)$ has only a finite number of poles. (Hint: you might start by considering the function $g(z) = 1/f(1/z)$. What is the behaviour of $g(z)$ near 0?)

(b) Show that $f(z)$ is a ratio of two polynomials: there are polynomials $P(z)$, $Q(z)$ so that

$$f(z) = \frac{P(z)}{Q(z)}.$$

(Such a ratio of two polynomials is called a *rational function*.)

(Proof omitted, since we discussed this in class.)

(6) When does $\sum_{n=1}^{\infty} (-1)^n \frac{n \sin bn}{a^2 - n^2}$ converge, with $a, b \in \mathbb{R}$, $a \notin \mathbb{Z}$? Find the value when it does converge.

Since the summand is symmetric about $n = 0$, we have

$$\sum_{n=1}^{\infty} (-1)^n \frac{n \sin bn}{a^2 - n^2} = \frac{1}{2} \sum_{n=-\infty}^{\infty} (-1)^n \frac{n \sin bn}{a^2 - n^2}.$$

Now consider the function

$$f(z) = \frac{z \sin bz}{a^2 - z^2} \frac{\pi}{\sin \pi z}.$$

In addition to the poles at $z = \pm a$, this has a pole at each integer n with residue

$$(-1)^n n \sin bn / (a^2 - n^2).$$

Therefore,

$$\frac{1}{2\pi i} \oint_{\Gamma_N} f(z) dz = \sum_{n=-N}^N (-1)^n \frac{n \sin bn}{a^2 - n^2} + \text{Res}_{z=a} f(z) + \text{Res}_{z=-a} f(z)$$

for a suitable path Γ_N enclosing $-N, \dots, N$ and a and $-a$. If we can show that the integral along Γ_N of $f(z)$ tends to 0 as $N \rightarrow \infty$, then the sum will converge, and we will be able to evaluate it in terms of the residues of $f(z)$. Let us first temporarily give names to these integrals and their limit (if the integral exists):

$$\begin{aligned}T_N &= \frac{1}{2\pi i} \oint_{\Gamma_N} f(z) dz \\ T &= \lim_{N \rightarrow \infty} T_N.\end{aligned}$$

Then we have

$$\begin{aligned} \sum_{n=1}^{\infty} (-1)^n \frac{n \sin bn}{a^2 - n^2} &= \frac{T}{2} - \frac{1}{2} \operatorname{Res}_{z=a} f(z) - \frac{1}{2} \operatorname{Res}_{z=-a} f(z) \\ &= \frac{T}{2} + \frac{\pi \sin ba}{\sin \pi a} \end{aligned}$$

Let's first consider the asymptotic behaviour of $f(z)$ away from the real axis.

Lemma 1. *If $z = x + iy$ with $|y| > \epsilon > 0$, then there are constants K so that*

$$|f(z)| \leq Ky e^{(|b|-\pi)|y|}.$$

Proof. For $z = x + iy$ with $y > \epsilon > 0$, we have

$$\begin{aligned} |\sin z| &= \left| \frac{e^{ix-y} - e^{ix+y}}{2i} \right| \\ &< e^y \\ &\geq \frac{e^y - e^{-y}}{2} \geq (1 - e^{-2\epsilon}) e^y. \end{aligned}$$

and so, for a suitable constant K_1 and $y > \epsilon$,

$$K_1 e^y < \sin(x + iy) < e^y.$$

Applying this to the function $\sin bz \wedge \sin \pi z$, we see that for suitable constants

$$\begin{aligned} |\sin bz| &\leq K_2 e^{|b|y} \\ \left| \frac{1}{\sin \pi z} \right| &\leq K_3 e^{-\pi y} \\ |z| &\leq y \\ \left| \frac{1}{z^2 - a^2} \right| &\leq K_4 \\ |f(z)| &= \left| \frac{z \sin bz}{(z^2 - a^2) \sin \pi z} \right| \leq K_5 y e^{(|b|-\pi)y}. \end{aligned}$$

The case for $\operatorname{Im} y < 0$ follows since the function $f(z)$ is odd. □

For the integral along Γ_N to go to zero, this function $f(z)$ should at least go to 0 as we increase y ; this happens when $|b| < \pi$. So let us suppose for the moment that $-\pi < b < \pi$, and let's show that the limit is 0 in this case.

Lemma 2. *For $-\pi < b < \pi$,*

$$\lim_{N \rightarrow \infty} T_N = 0.$$

Proof. Take Γ_N to be a rectangle of height $h = \sqrt{N}$ and width $2N + 1$. Then along the two horizontal sides I_1 and I_3 and N sufficiently large, we have, by the M-L theorem,

$$\left| \oint_{I_1+I_3} f(z) dz \right| \leq 2(2N + 1) K_5 h e^{(|b|-\pi)h} \leq K_6 N^{3/2} e^{(|b|-\pi)\sqrt{N}}$$

which goes to 0 as $N \rightarrow \infty$.

Along the vertical sides I_2 and I_4 , $|y| \leq h$ and, for N sufficiently large (say $N > 2|a|$), we have

$$\begin{aligned} |\sin \pi z| &= \left| \frac{e^{i\pi(N+1/2+iy)} - e^{i\pi(N+1/2-iy)}}{2i} \right| \\ &= \left| \frac{ie^{-y} + ie^y}{2i} \right| \\ &\geq e^{-|y|} \\ \frac{1}{\sin \pi z} &\geq e^{|y|} \\ |f(z)| &= \left| \frac{z}{z^2 - a^2} \frac{\sin bz}{\sin \pi z} \right| \\ &\leq \frac{K_7}{N} e^{(|b|-\pi)y} \\ &\leq \frac{K_7}{N}. \end{aligned}$$

and so by the M-L theorem

$$\left| \oint_{I_2+I_4} f(z) dz \right| \leq \frac{K_7}{N} 4h \leq \frac{4K_7}{\sqrt{N}}$$

which again goes to 0 as $N \rightarrow \infty$. Thus, for $-\pi < b < \pi$,

$$\lim_{N \rightarrow \infty} T_N = 0.$$

□

Now let's consider the case for general b . Note that if b is an integer multiple of π , then $\sin bn = 0$ for all integers n and the summands are identically 0. On the other hand, for n an integer, $\sin(bn) = \sin(b \pm 2\pi)n$ and so the values of b differing by multiples of 2π give the same sum. This lets us evaluate the sum for any value of b which is not of the form $b = (2N+1)\pi$.

In summary, the value of the sum is

$$\sum_{n=1}^{\infty} (-1)^n \frac{n \sin(bn)}{a^2 - n^2} = \frac{\pi \sin \tilde{b}a}{\sin \pi a}$$

where $\tilde{b} = b \bmod 2\pi$ takes values in the interval $-\pi < \tilde{b} \leq \pi$.

- (7) (For your amusement; only minimal extra credit will be provided, so only work on this if you have extra time!) Suppose that the power series

$$f(z) = \sum_{n=0}^{\infty} c_n z^n$$

has radius of convergence R , with $0 < R < \infty$. Suppose that $f(z)$ has an extension which has a pole (as opposed to any other kind of singularity) on the boundary of the disk of convergence. Show that the series above diverges at all points on the boundary of the disk of convergence.

Suppose that $f(z)$ has a pole at $z = z_0$, with $|z_0| = R$. For convenience, let's rescale f so that the pole is at $z = 1$: consider

$$g(z) = f(z_0 z) = \sum_{n=0}^{\infty} a_n z^n.$$

It will suffice to show that the power series for $g(z)$ does not converge anywhere on the circle $|z| = 1$. Suppose not: if there is sum z , $|z| = 1$, so that the sum converges, then in particular the terms of the series go to zero:

$$0 = \lim_{n \rightarrow \infty} |a_n z^n| = \lim_{n \rightarrow \infty} |a_n|.$$

Pick an $\epsilon > 0$, and choose N so that $|a_n| < \epsilon$ for $n \geq N$. Now consider the function $(1-x)g(x)$, for real x , $0 < x < 1$, and its limit as $x \rightarrow 1$. (That is, consider the radial limit of $(1-z)g(z)$ as we approach the pole of g at $z = 1$.)

$$\begin{aligned} \left| \lim_{x \rightarrow 1} (1-x)g(x) \right| &= \left| \lim_{x \rightarrow 1} (1-x) \sum_{n=0}^{\infty} a_n x^n \right| \\ &\leq \left| \lim_{x \rightarrow 1} (1-x) \sum_{n=0}^{N-1} a_n x^n \right| + \lim_{x \rightarrow 1} (1-x) \sum_{n=N}^{\infty} |a_n| x^n \\ &\leq 0 + \lim_{x \rightarrow 1} (1-x) \sum_{n=N}^{\infty} \epsilon x^n \\ &= \lim_{x \rightarrow 1} (1-x) \frac{\epsilon x^N}{1-x} \\ &= \epsilon. \end{aligned}$$

Since ϵ was arbitrary, we actually have

$$\lim_{x \rightarrow 1} (1-x)g(x) = 0.$$

But this contradicts the given condition that g has a pole at 1: by definition, this means that there is some $n > 0$ and an analytic function $h(z)$ defined near $z = 1$, with $h(1) \neq 0$, so that

$$g(z) = \frac{h(z)}{(z-1)^n}$$

so that

$$\lim_{z \rightarrow 1} (1-z)g(z) = \begin{cases} -h(1) & \text{if } n=1, \\ \infty & \text{if } n > 1. \end{cases}$$

In either case,

$$\lim_{z \rightarrow 1} (1-z)g(z) \neq 0$$

contradicting our earlier assumption that the power series for $g(z)$ converged at some point on the boundary.