FINAL PAPER OPTION

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As announced on the syllabus, you may choose to do a final paper instead of a final exam. The paper should cover some topic not covered in the course, but related to it. It is not expected to be original research, but should show good understanding of the material and should draw from more than one source.

Some guidelines for the papers:

- Cover a topic well. Depending on your mathematical writing style, this could take 5 pages, or it could be 20. (You can see this from our two textbooks: Needham is much more verbose than Bak and Newman.) Note that it is not necessarily easier to write a short paper: it takes good judgement to distill a subject to its essence.
- Pick a small, focussed topic: the smaller your topic, the more you'll be able to master it fully before writing about it.
- Write something that you would enjoy reading.
- You can aim your paper for someone who has had a first course on complex analysis, or someone who has not, but be consistent in your choice.
- Make sure you understand what you write about. If you don't quite understand some part of the topic, it's usually better to omit it from your paper.

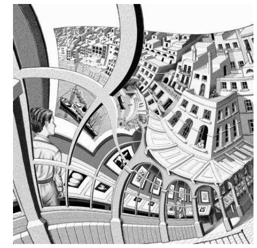
The paper will satisfy the Junior Paper requirement, subject to approval by the Head Tutor (Professor Taubes).

The paper is due on January 16, the last day of reading period. There will be a system for exchanging and commenting on drafts during reading period; you should have a first draft ready by the beginning of reading period.

One good source for topics is to take something mentioned in class and explore it further. A few other possible topics are listed below. Several of them are a bit too large in scope; you made need to work to limit the scope a little. These are only suggestions to get you started thinking; you should feel free to come up with your own topic! In any case, please come talk to me for further references or suggestions on your topics.

Conformal maps of the brain: As mentioned on the first day of class, complex analysis has recently been used to construct conformal maps of the brain for use in medicine. This relies on a theorem, the Uniformization Theorem, which we will not prove in class, but which is related to the Riemann Mapping Theorem, which we will prove. Explain the Uniformization Theorem and either give a proof or the outline of a proof, or explain how to compute such maps in practice.

Artistic conformal mapping: Conformal mappings can also be used to transform paintings with minimal distortion. This was used (implicitly) by Escher in the painting "Print Gallery", as you can see on the right. This is nicely explained at `http://escherdroste.math.leidenuniv.nl/`. See if you can find another example of conformal mappings or complex analysis in art (or create your own work of art!) and explain the mathematics behind it. There are several other works of Escher that would make a good starting point.



Orbits in central force field: A particle moving in a central force field with force $-1/r^2$ (i.e., according to Newtonian gravity) traces out an elliptical orbit, with one focus of the ellipse at the origin of the force field.One unusual proof of this fact uses complex analysis: the map $z \mapsto z^2$ takes orbits for the central force field $-r$ (i.e., a simple harmonic oscillator in two dimensions, whose orbits are ellipses with center at the origin) to orbits for the field $-1/r^2$. There is a discussion of this in Needham, Chapter 5.X, and in the reference mention therein. Come to your own understanding of this fact and explain it.

Taylor series and Fourier analysis: The Taylor series we have been exploiting in our study of complex analytic functions are intimately connected with Fourier series when we look at the values of the Taylor series on the unit circle. Show how to use this connection to calculate Fourier series for some common functions. Section 2.III.7 (pages 77–79) in Needham is a good starting point, but you should find your own examples.

Elliptic functions: One of the main subjects in 19th century mathematics was elliptic functions, which can be thought of as generalizations of trigonometric functions; where the trigonometric functions are periodic with period 2π, the elliptic functions have two independent periods $p_1, p_2 \in \mathbb{C}$, and so take the same values in a lattice in the plane. Either explain some of the geometry of elliptic functions and how they relate to complex tori, or explain how to use them to do practical integrals.

Brownian motion and percolation: There are a number of 2-dimensional physical models that have recently been shown to be conformally invariant, including Brownian motion (a.k.a. random walks in the plane) and percolation. Show how knowing that a model is conformally invariant allows you to compute physical quantities in practice. (The actual proofs of conformal invariance are probably too hard, except possibly for Brownian motion.)

Practical conformal mapping: Explain how to construct explicit conformal maps between a variety of different regions. Draw pictures of the results.

Branched coverings and Riemann surfaces: The branches of a multi-valued function can be glued together to construct a *Riemann surface*: a surface that "locally looks like" $\mathbb{C}$. Learn about and explain this connection.