Math 113 Final Exam

Due January 16, 2004, at 5:00 PM

This exam is open notes and open book: you may use any of your notes and the textbooks for this course. You may also use a reference on real analysis if you wish. (Please indicate on the exam if you do so.) No other aids are permitted, and you may not discuss the content of the exam with anyone.

Each problem is worth 20 points. Your best 5 out of the 6 problems will count.

There will be a set of hints posted at 10PM today (Wednesday). Which problems get hints will be determined in part by you, the students; please e-mail me to nominate which problems you don't know how to get started on, or where else you got stuck.

1. Consider the function
$$f(z) = \frac{1}{1-z-z^2} = \sum_{n=0}^{\infty} c_n z^n.$$
   (a) Show that $c_{n+2} = c_{n+1} + c_n$ for $n \geq 0$. (The sequence c_n is called the *Fibonacci sequence.*)
   (b) What is the radius of convergence of the series?
   (c) Deduce a consequence of your answer in (b) for the Fibonacci sequence.
   (d) (10 points) Consider the series a_n defined by
$$\begin{aligned} a_0 &= 1 \\ a_n &= \frac{a_{n-1}}{1} + \frac{a_{n-2}}{2} + \cdots + \frac{a_0}{n}, \qquad n > 0 \end{aligned}$$
   Find $\lim_{n\to\infty} \frac{a_n}{a_{n-1}}$.

2. Let D be the unit disk $|z| < 1$ and let $f : D \to \mathbb{C}$ be an analytic function on D with a continuous extension to the boundary, and suppose that f is not zero on the boundary. Show that
$$\max_{z\in D} \operatorname{Re}\left(\frac{zf'(z)}{f(z)}\right) \geq \#\text{ of zeros of } f \text{ in } D.$$

3. Evaluate
$$\int_0^{\infty} \frac{\cos ax}{\cosh x}\, dx.$$

4. Let $f : D \to \mathbb{C}$ be a function on the unit disk which satisfies $|f(z)| < 1$. Show that if $f(\alpha) = 0$, then

$$|f(z)| \leq \left| \frac{z - \alpha}{\bar{\alpha} z - 1} \right|.$$

5. A function $f : \mathbb{C} \to \mathbb{C}$ is said to be *quasiperiodic* with period ω if there are constants $a, b \in \mathbb{C}$ so that

$$f(z + \omega) = e^{az+b} f(z).$$

In this problem we will consider doubly quasiperiodic functions, with two periods ω_1 and ω_2 (with $\operatorname{Im} \omega_2/\omega_1 > 0$, so that ω_1 and ω_2 span a lattice), satisfying

$$f(z + \omega_1) = e^{a_1 z + b_2} f(z)$$
$$f(z + \omega_2) = e^{a_2 z + b_2} f(z).$$

(a) Find all entire, doubly quasiperiodic functions which are never zero.

(b) If $f(z)$ is entire and doubly quasiperiodic and has a simple zero at $n_1\omega_1 + n_2\omega_2$ for all integers n_1, n_2 and no other zeros, show that

$$a_1\omega_2 - a_2\omega_1 = 2\pi i.$$

6. Let $h : D \setminus \{0\} \to \mathbb{R}$ be a harmonic function on the punctured unit disk. Suppose that

$$\lim_{z \to 0} z h(z) = 0.$$

Show that

$$h(z) = \alpha \log(|z|) + g(z)$$

for a suitable $\alpha \in \mathbb{R}$ and harmonic function $g : D \to \mathbb{R}$ defined on the whole unit disk (including $z = 0$).