WHY THE RESIDUE?

MATHEMATICS 113

October 23, 2003

On Wednesday (October 22), we had a class discussion about the residue theorem and why it is true (as opposed to a proof that it is true). There were a lot of good reasons given, but they came somewhat out of order; this is my attempt to organize them into a logical order according to which question they try to answer.

Some of these questions will seem completely intuitive and not in need of an explanation to some of you, but I wanted to be as complete as possible. It is my hope that you will gain an intuition for analytic functions so that all of the questions seem obvious!

Please let me know if I missed anything, or if you come up with some new answers to these questions that aren't listed!

Question 1. *Why is the residue theorem true? For a simple closed curve* Γ *and analytic function* $f(z)$ *with isolated singularities, why is*

$$\oint_\Gamma f(z)\,dz = \sum_{\substack{\text{singularity} \\ a \text{ of } f}} 2\pi i \times n(\Gamma, a) \times \mathrm{Res}_a(f)?$$

There were two basic approaches to this question suggested.

Reason 1.1. To evaluate a curve along an arbitrary closed curve, possibly enclosing some isolated singularities, we can shrink the curve by deformations until it becomes a union of small circles around each singularity and straight lines [connecting the circles to some base point]. The integrals along the straight lines cancel out [since each line is traversed twice in opposite directions], and the integral around a small circle picks out the $1/z$ coefficient at that singularity, times $2\pi i$.

Of course, this reason naturally raises two other questions: "Why can we deform the curve?" (Question 2) and "Why does the integral around a small circle pick out the $1/z$ coefficient?" (Question 4).

Reason 1.2. The integral is a *linear* function of the function $f(z)$, so we can split $f(z)$ into an analytic function $g(z)$ and a principal part for each singularity. The integral of the analytic function $g(z)$ vanishes, and the integral of a function with a single singularity picks out the residue at the singularity times the winding number.

Again, we have two subsidiary questions: "Why is the integral of an analytic function $g(z)$ on a closed curve zero?" (Question 3) and, again, why we get the residue in particular (Question 4).

Question 2. *Why is the integral along a complex contour invariant under deformations that fix the endpoints (and that don't pass over any singularities)?*

Reason 2.1. If we deform a smooth curve a little bit and superimpose the new curve on the old curve, we generally see some crescent-shaped regions cut out by the two curves. The difference between the integral of $f(z)$ along the two curves will be the sum of the integrals along the boundary of these crescent-shaped regions; but the integral of an analytic function $f(z)$, defined on a simply-connected region R, along the boundary of R is equal to zero.

Reason 2.2. We have seen that complex differentiable functions are incredibly conservative, and have all sorts of nice properties. In particular, once differentiable functions are both infinitely differentiable, and have an anti-derivative (at least in small regions): if $f(z)$ is analytic in a disk, then there is a function $F(z)$ so that

$$f(z) = F'(z)$$

But the Fundamental Theorem of Calculus [which is also true in the complex setting] says that

$$\int_a^b f(z)\,dz = F(b) - F(a).$$

In particular, the integral does not depend on the path we take to get from a to b, as long as we stay within a region for which $F(z)$ is defined.

Question 3. *Why is the integral of an analytic function $f(z)$ along a simple closed curve Γ equal to 0 (as long as $f(z)$ is analytic in the whole interior of Γ)?*

Reason 3.1. The interior of Γ is a disk, which is simply-connected, so we can find an anti-derivative $F(z)$ to the function f defined on the interior of Γ as well as Γ itself. But then, by the Fundamental Theorem of Calculus (as in Reason 2.2), if Γ runs from the point z back to z,

$$\oint_\Gamma f(z)\,dz = F(z) - F(z) = 0.$$

For all the above reasons, we reduce the original question to an integral around a single singularity. For simplicity, let's suppose that we're integrating around a circle C of radius 1 that winds once counterclockwise around a singularity at 0.

Question 4. *If $f(z)$ has an isolated singularity at 0, why is*

$$\oint_C f(z)\,dz = 2\pi i \operatorname{Res}_0(f)?$$

That is, why does the integral pick up $2\pi i$ times the coefficient of $1/z$?

By linearity of the integral, we can reduce this question to the integral of a simple power:

Question 5. *Why is $\oint_C z^k\,dz$ equal to 0 if $k \neq -1$, and equal to $2\pi i$ if $k = -1$?*

We came up with many different answers to this question, addressing different possible values of k.

Reason 5.1. For $k \geq 0$, z^k is analytic on the disk, and so the integral around a closed curve is 0 as in Question 3.

Reason 5.2. For $k \neq -1$, z^k has an antiderivative defined on $\mathbb{C} \setminus \{0\}$, namely $\frac{1}{k+1} z^{k+1}$, and so the integral of z^k around a closed curve is 0 by the Fundamental Theorem of Calculus, as in Reason 3.1.

Reason 5.3. For $k = -1$, as we integrate around the closed circle, the angle of dz is $\theta + \pi/2$ and the angle of z^{-1} is $-\theta$; adding these up, we find that $z^{-1}\,dz$ has a constant angle of $\pi/2$, so we don't get any cancellation and the integral is at angle $\pi/2$, i.e., some multiple of i. For $k \neq -1$, the angle of z^k is $k\theta$ so the angle of $z^k\,dz$ is $(k+1)\theta + \pi/2$, which rotates as we go around the circle (with θ increasing from 0 to 2π). Since the magnitude of the vectors we add remains constant, the integral is 0 for $k \neq -1$.

Reason 5.4. For $k = -1$, we expect that

$$\int_a^b \frac{dz}{z} = \log z\big|_a^b$$

The logarithm function is the inverse of the exponential function, and so if we write z in polar coordinates, we have

$$\log re^{i\theta} = \log r + i\theta$$

However, the angle θ is only defined up to adding a multiple of 2π, and if we move around the circle counterclockwise starting at 0, the angle

θ increases continuously from 0 up to 2π, so

$$\oint_c \frac{dz}{z} = \log z|_C = \log(e^{2\pi i}) - log(e^0) = 2\pi i.$$

(The notation is a little loose here, but that's OK, since we're giving a reason, not a proof.)

Reason 5.5. For $k = -1$, $\oint \frac{dz}{z}$, consider taking the original circle and shrinking it down until it has radius r. The length of the curve we integrate along scales down by a factor of r, while the function $\frac{1}{z}$ which we are integrating scales up by a factor of $\frac{1}{r}$. These two cancel out, so the integral seems to stay constant, and it is reasonable to suppose that the magnitude of the integral is equal to the length of the original curve, which is 2π.

Reason 5.6. For $k \geq 0$, if we shrink our original contour to a circle of small radius r, the length of the curve scales down by a factor of r while the value of the function z^k scales down by a factor of r^k (or remains constant, if $k = 0$). By the M-L theorem, the total integral is bounded above by some constant times r^{k+1}; since r can be as small as we like, the integral must be 0.

Reason 5.7. For $k = -l \leq -2$, if we expand the original contour to a large circle of radius R, the length of the contour scales up by a factor of R while the function $1/z^l$ scales down by a factor of R^{-l}. The scaling down of $1/z^l$ beats the scaling up of the length of the curve, so by the M-L theorem the integral is bounded above by a constant times R^{1-l}, which can be made as small as we like for R sufficiently large.

Reason 5.8. For $k = -2$, consider perturbing the function a little bit:

$$\frac{1}{z^2} = \lim_{\epsilon \to 0} \frac{1}{z^2 - \epsilon^2} = \lim_{\epsilon \to 0} \frac{1}{2\epsilon} \left(\frac{1}{z - \epsilon} - \frac{1}{z + \epsilon} \right)$$

(The second step is just the partial fraction expansion of $\frac{1}{z^2 - \epsilon^2}$.) The perturbed function has a simple pole with residue $\frac{1}{2\epsilon}$ at $z = \epsilon$ and a simple pole with residue $-\frac{1}{2\epsilon}$ at $z = -\epsilon$. For ϵ sufficiently small, both poles will be contained inside our circle C, but since the two residues have equal magnitude but opposite signs, the integral around any loop that encloses both poles will be 0. (This argument is closely related to the electric field of a dipole, written in terms of the fields of two close and opposite-charged particles. A similar argument works for any $k \leq -2$.)