Solutions for Problem Set #2
due September 26, 2003
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(B&N 2.2) *By comparing coefficients or by use of the Cauchy-Riemann equations, determine which of the following polynomials are analytic. (5 points each)*

a. $P(x+iy) = x^3 - 3xy^2 - x + i(3x^2y - y^3 - y)$

We require that the partial derivatives are related by $P_y = iP_x$. We compute:

$$\begin{aligned} P_x &= 3x^2 - 3y^2 - 1 + i(6xy) \\ iP_x &= i(3x^2 - 3y^2 - 1) - 6xy \\ P_y &= -6xy + i(3x^2 - 3y^2 - 1) \end{aligned}$$

So the last two equations are equal, so P is analytic.

b. $P(x+iy) = x^2 + iy^2$

We do the same thing as in part a:

$$\begin{aligned} P_x &= 2x \\ iP_x &= 2ix \\ P_y &= 2iy \end{aligned}$$

The last two equations are not equal, so P is not analytic.

c. $P(x+iy) = 2xy + i(y^2 - x^2)$

$$\begin{aligned} P_x &= 2y - 2ix \\ iP_x &= 2iy + 2x \\ P_y &= 2x + 2iy \end{aligned}$$

So P is analytic.

(B&N 2.3) *Show that no nonconstant analytic polynomial can take imaginary values only.*

Suppose P is an nonconstant analytic polynomial, and $P(x+iy)$ is always purely imaginary. Therefore, the partials P_x and P_y must also be purely imaginary. But, then iP_x is a real number, so $P_y = iP_x$ must be zero. If both partials are zero, then P must be a constant function. This is a contradiction.

(B&N 2.5) *Prove that if f and g are differentiable at z, then the following functions are differentiable with the derivatives shown: (5 points each)*

a. $(f+g)'(z) = f'(z) + g'(z)$

$$\begin{aligned}
(f+g)'(z) &= \lim_{h\to 0} \frac{f(z+h)+g(z+h)-(f(z)+g(z))}{h} \\
&= \lim_{h\to 0} \frac{f(z+h)-f(z)}{h} + \lim_{h\to 0} \frac{g(z+h)-g(z)}{h} \\
&= f'(z) + g'(z)
\end{aligned}$$

b. $(fg)'(z) = f'(z)g(z) + f(z)g'(z)$

$$\begin{aligned}
(fg)'(z) &= \lim_{h\to 0} \frac{fg(z+h)-fg(z)}{h} \\
&= \lim_{h\to 0} \frac{fg(z+h)-f(z)g(z+h)+f(z)g(z+h)-fg(z)}{h} \\
&= \lim_{h\to 0} g(z+h)\frac{f(z+h)-f(z)}{h} \\
&\quad + \lim_{h\to 0} f(z)\frac{g(z+h)-g(z)}{h} \\
&= g(z)f'(z) + f(z)g'(z)
\end{aligned}$$

Notice that the last step uses the continuity of g (implied because it is differentiable) to show that $\lim_{h\to 0} g(z+h) = g(z)$.

c. $\left(\frac{f}{g}\right)'(z) = \frac{f'(z)g(z)-f(z)g'(z)}{g^2(z)}$ *(assuming that $g(z) \neq 0$)*

Consider just the function $1/g$, and then we will use the product rule:

$$\begin{aligned}
\left(\frac{1}{g}\right)(z) &= \lim_{h\to 0} \frac{1/g(z+h) - 1/g(z)}{h} \\
&= \lim_{h\to 0} \frac{1}{g(z)g(z+h)} \frac{g(z) - g(z+h)}{h} \\
&= -\frac{1}{g^2(z)} g'(z)
\end{aligned}$$

Now apply the product rule, but with $1/g$ instead of g:

$$\begin{aligned}
\left(f\frac{1}{g}\right)(z) &= f'(z)\frac{1}{g} + f(z)\left(-\frac{1}{g^2(z)}g'(z)\right) \\
&= \frac{f'(z)g(z) - f(z)g'(z)}{g^2(z)}
\end{aligned}$$

(B&N 2.6) *Prove that $P(z) = a_0 + a_1 z + \ldots + a_n z^n$ is differentiable and has derivative $P'(z) = a_1 + 2a_2 z + \ldots + na_n z^{n-1}$. (5 points)*

First let us deal with the case of a monic monomial: $f(z) = z^j$. I claim that the derivative of f is jz^{j-1}. We will prove this by induction. For $j = 0$, f is constant, so $f'(z) = 0$. For $j = 1$,

$$\begin{aligned}
f'(z) &= \lim_{h\to 0} \frac{(z+h) - z}{h} \\
&= \lim_{h\to 0} \frac{h}{h} \\
&= 1 = 1z^0
\end{aligned}$$

Now for $j > 1$, $f(z) = z^j = z^{j-1}z$, so applying the product rule and the inductive hypothesis,

$$\begin{aligned}
f'(z) &= (j-1)z^{j-2}z + z^{j-1}(1) \\
&= (j-1)z^{j-1} + z^{j-1} \\
&= jz^{j-1}
\end{aligned}$$

By applying the product rule, we find that the derivative of $a_j z^j$ is $ja_j z^j + 0z^j = ja_j z^j$. Therefore, if $P(z)$ is as in the problem statement, then by the summation rule, $P'(z)$ exists and is equal to $a_1 + 2a_2 z + \ldots + na_n z^{n-1}$.

(B&N 9) *Suppose* $\sum c_n z^n$ *has a radius of convergence* R*. Find the radius of convergence* R' *of: (5 points each)*

a. $\sum n^p c_n z^n$

We use the root test for power series:

$$\begin{aligned} 1/R' &= \limsup_{n\to\infty} \sqrt[n]{|n^p c_n|} \\ &= (\limsup_{n\to\infty} \sqrt[n]{|n|})^p (\limsup_{n\to\infty} \sqrt[n]{|c_n|}) \\ &= 1^p \cdot 1/R \\ &= 1/R \end{aligned}$$

Note that we used the fact from real analysis that $\lim_{n\to\infty} \sqrt[n]{n} = 0$. This can be checked by L'Hopital's rule.

Therefore $R' = R$.

b. $\sum |c_n| z^n$

Using the same test:

$$\begin{aligned} 1/R' &= \limsup_{n\to\infty} \sqrt[n]{||c_n||} \\ &= \limsup_{n\to\infty} \sqrt[n]{|c_n|} \\ &= 1/R \end{aligned}$$

so, again, $R' = R$.

c. $\sum c_n^2 z^n$

$$\begin{aligned} 1/R' &= \limsup_{n\to\infty} \sqrt[n]{c_n^2} \\ &= (\limsup_{n\to\infty} \sqrt[n]{c_n})^2 \\ &= 1/R^2 \end{aligned}$$

so $R' = R^2$.

(B&N 11) *Show that* $\sum_{n=1}^{\infty}(z^n/n)$ *converges at all points on the unit circle except at* $z = 1$ *(5 points, plus 5 points extra credit for proving Dirichlet's Test or the equivalent)*

When $z = 1$, the sum becomes $\sum 1/n$, which diverges from basic real analysis.

Dirichlet's test states that if b_n is a sequence (of real or complex number) such that the partial sums are bounded, and a_n is a decreasing sequence (of real numbers), which goes to zero as n goes to infinity, then the infinite series $\sum a_n b_n$ converges. We apply this to our sum with $a_n = 1/n$, and $b_n = z^n$.

That the sequence a_n meets the conditions for the test is obvious. To show that the partial sums of the b_n are bounded, we just use the geometric series formula:

$$\begin{aligned} \left|\sum_{i=1}^{n} a_i\right| &= \left|\sum_{i=1}^{n} z^i\right| \\ &= \left|z\frac{z^n - 1}{z - 1}\right| \\ &\leq \frac{2|z|}{|z-1|} \end{aligned}$$

so it is bounded for $z \neq 1$. Therefore, by Dirichlet's test, the power series converges on the unit disk so long as $z \neq 1$.

To prove Dirichlet's Test (as extra credit), we define $A_n = \sum_{i=0}^{n} a_n$, let M be the bound for the A_n, then the difference between the nth and mth partial sums of $a_n b_n$ is:

$$\begin{aligned} \sum_{i=m}^{n} a_i b_i &= \sum_{i=m}^{n} (A_{i+1} - A_i) b_i \\ &= \sum_{i=m+1}^{n} A_i(b_{i-1} - b_i) + A_{m+1}b_m - A_n b_n \\ \left|\sum_{i=m}^{n} a_i b_i\right| &\leq \sum_{i=m+1}^{n} |A_i(b_{i-1} - b_i)| + |A_{m+1}b_m| + |A_n b_n| \\ &\leq M \sum_{i=m+1}^{n} (b_{i-1} - b_i) + Mb_m + Mb_n \end{aligned}$$

because the sequence (b_n) is monotonically decreasing, and so,

$$\begin{aligned} &\leq M(b_m - b_n + b_m + b_n) \\ &\leq 2Mb_m \end{aligned}$$

Since b_m goes to zero, then the difference between partials must go to zero as m gets large. Therefore the sequence $(a_n b_n)$ is Cauchy, and therefore it converges.

(Needham 4.2) *The mapping $z \mapsto z^3$ acts on an infinitesimal shape and the image is examined. It is found that the shape has been rotated by π, and its linear dimensions expanded by 12. Where was the shape originally located? [There are two possibilities] (5 points)*

The transformation of an infinitesimal shape corresponds to the derivative. In this case, the derivative must be $12 \operatorname{cis} \pi = -12$. Since the derivative is $3z^2$. Therefore,

$$\begin{aligned} 3z^2 &= -12 \\ z^2 &= -4 \\ z &= \pm 2i \end{aligned}$$

(Needham 4.5) *Consider $f(x+iy) = (x^2+y^2)+i(y/x)$. Find and sketch the curves that are mapped by f into (a) horizontal lines, and (b) vertical lines. Notice from your answers that f appears to be conformal. Show that it is not in two ways: (i) by explicitly finding some curves whose angle of intersection isn't preserved; and (ii) by using the Cauchy-Riemann equations. (10 points)*

Suppose $f(x+iy) = (x^2+y^2) + i(y/x) = a + bi$. The inverse image of a horizontal line are the points where $b = y/x$ is constant. These are the points that satisfy $y = xb$, which are straight lines through the origin. The inverse image of a vertical line is all the points where $a = x^2+y^2$ is constant. These are circles with radius $\sqrt{a}$. These curves are clearly perpendicular.

Consider the two lines $1 + yi$ and $x + i$, for x, y real. The images of these lines are $(1 + y^2) + yi$, and $(x^2 + 1) + i/x$. Now, let us solve for a function giving the real part for the imaginary part. The first one is defined by the points where $a = 1 + b^2$, and the second is defined by $a = 1 + b^{-2}$. In order to compute the tangent lines, we take the derivatives, and get $a = 2b$ and $a = -2b^{-3}$ respectively. Since the intersection is when $x = y = 1$, then

$b = 1$, so the slopes at the intersection are 2 and -2 respectively These are not perpendicular, because there product is not -4.

If we write $f(x+iy) = u(x,y)+iv(x,y)$, then the partial derivatives are:

$$
\begin{aligned}
u_x &= 2x \\
u_y &= 2y \\
v_x &= -y/x^2 \\
v_y &= 1/x
\end{aligned}
$$

These clearly do not satisfy either one of the Cauchy-Riemann equations.

(Needham 4.6) *Continuing from the previous exercise, show that no choice of v can make $f(x+iy) = (x^2+y^2)+iv$ analytic. (5 points extra credit)*

By the Cauchy-Riemann equations, f will be analytic iff:

$$
\begin{aligned}
v_x &= -u_y = -2y \\
v_y &= u_x = 2x
\end{aligned}
$$

By integrating the first, we conclude that v must be equal to $-2xy + C_1(x)$ for some function C_1 which depends only on x. By integrating the second, v must be $2xy + C_2(y)$ for some constant C_2, which likewise depends only on y. However, there is now way for these to be equal for all x, y, so no choice of v will make f analytic.

(B&N 3.4) *Suppose that g is a continuous "$\sqrt{z}$" (i.e., $g^2(z) = z$) in some neighborhood of z. Verify that $g'(z) = \frac{1}{2\sqrt{z}}$. (5 points extra credit)*

The book gives a hint, for doing the computation directly, but there is an easier way using the chain rule from the previous problem.

Define $h(z) = z^2$, and then we know that $h \circ g(z) = z$. Taking the derivative of this equation and applying the chain rule gives:

$$
\begin{aligned}
h'(g(z))g'(z) &= 1 \\
2g(z)g'(z) &= 1 \\
g'(z) &= \frac{1}{2g(z)}
\end{aligned}
$$

which is what we want.