Math 113 Mid-term exam

Due November 3, 2003, at 5:00 PM

This exam is open notes and open book: you may use any of your notes and the textbooks for this course. You may also use a reference on real analysis if you wish. No other aids are permitted, and you may not discuss the content of the exam with anyone.

1. Consider the map $f(z) = 1/z$.

   (a) For which points $z \in \mathbb{C}$ does f shrink a sufficiently small neighborhood of z? That is, for which points z is

   $$\lim_{\epsilon \to 0} \frac{\text{Area}(f(D_\epsilon(z)))}{\text{Area}(D_\epsilon(z))} < 1?$$

   You need not prove your answer.

   (b) If $z = x + iy$, what are the images of the lines $x = \text{const}$ and the lines $y = \text{const}$? Draw the images and identify the geometric objects they trace out, with proof.

   (c) What is the image of the first quadrant? Where does the boundary of the first quadrant go (with orientation)?

2. Evaluate the following sums and integrals:

   (a) $\displaystyle\int_0^{2\pi} \frac{d\theta}{1 - 2a\cos\theta + a^2}$ (with $a \in \mathbb{C}$ and $|a| \neq 1$);

   (b) $\displaystyle\int_{-\infty}^{\infty} \frac{dx}{x^4 + 1}$

   (c) $\displaystyle\int_{-\infty}^{\infty} \frac{\cos ax}{x^2 + b^2} dx$ (with $a, b \in \mathbb{R}$);

   (d) $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)^3}$ (Hint: the function $\dfrac{\pi}{\sin \pi z}$ is useful for summing alternating series).

3. Find the doubly-infinite expansion for the function $f(z) = \frac{1}{(2z-1)(z-2)}$ which converges inside the annulus $\frac{1}{2} < |z| < 2$.

4. Find $\oint \bar{z}\, dz$ around

   (a) The boundary of an arbitrary rectangle and

   (b) The boundary of an arbitrary circle.

   Does the result have a geometric significance? (Hint: you might want to show that the integral is invariant under translation of the contour first.)

5. Suppose that $f(z)$ is a meromorphic function on all of $\mathbb{C}$, and that $\lim_{z\to\infty} f(z) = \infty$. (Specifically, this means that for every $r > 0$, there is an $R > 0$ so that for $|z| > R$, $|f(z)| > r$.)

   (a) Show that $f(z)$ has only a finite number of poles. (Hint: you might start by considering the function $g(z) = 1/f(1/z)$. What is the behaviour of $g(z)$ near 0?)

   (b) Show that $f(z)$ is a ratio of two polynomials: there are polynomials $P(z)$, $Q(z)$ so that
   $$f(z) = \frac{P(z)}{Q(z)}.$$
   (Such a ratio of two polynomials is called a *rational function*.)

6. When does $\displaystyle\sum_{n=1}^{\infty} (-1)^n \frac{n \sin bn}{a^2 - n^2}$ converge, with $a, b \in \mathbb{R}$, $a \notin \mathbb{Z}$? Find the value when it does converge.

7. (For your amusement; only minimal extra credit will be provided, so only work on this if you have extra time!) Suppose that the power series
   $$f(z) = \sum_{n=0}^{\infty} c_n z^n$$
   has radius of convergence R, with $0 < R < \infty$. Suppose that $f(z)$ has an extension which has a pole (as opposed to any other kind of singularity) on the boundary of the disk of convergence. Show that the series above diverges at all points on the boundary of the disk of convergence.