Mathematics 113 Fall 2002

HOMEWORK ASSIGNMENT # 3
DUE, Thursday, October 10

Collaboration: On the homework sets, collaboration is not only allowed but encouraged. However, you must write up and understand your own individual homework solutions, and you may not share written solutions. If you learn how to solve a problem by talking to a classmate, CA, or looking it up in a book, you should cite the source in your homework write-up, just as you would reference your sources in a literature or history class.
Show all your working, and write up your solutions as neatly as possible.

1. Let $f(z) = e^z$. Let a be a positive real number, and let C be the rectangle with vertices 0, a, $a + 2\pi i$ and $2\pi i$. Explicitly compute the integral
$$\oint_C f(z)dz$$
without using Cauchy's theorem, and verify that Cauchy's theorem applies in this case.

2. Let γ be the semicircular arc from 1 to -1 in the upper half plane. Use the ML-inequality (discussed in class, and stated as property 5 on page 93 of the text) to prove that
$$\left| \int_\gamma \frac{e^z}{z} \, dz \right| \leq \pi e$$

3. Let R be the region $\mathbb{C} \backslash \{[0, \infty]\}$. Let $f(z) = \sqrt{z}$, considered as a holomorphic function on R (in other words, after making a branch cut along the positive real axis), and such that $f(-1) = i$.

   (a) Let ϵ be a small real number, and let γ_ϵ be the path along the unit circle given explicitly by e^{it} for $t \in [\epsilon, 2\pi - \epsilon]$. Compute the integral
   $$I_\epsilon := \int_{\gamma_\epsilon} \sqrt{z} \, dz$$

   (b) Let I be the limit of this integral as ϵ approaches 0. Compute I, and explain why the fact that $I \neq 0$ does not contradict Cauchy's theorem.

   (c) Compute the real integral
   $$J := \int_0^1 \sqrt{x} \, dx.$$

   (d) Prove directly without computing I or J that $2J + I = 0$. (Hint: use Cauchy's theorem).

4. If R is a simply connected region with boundary C, prove that

$$A = \frac{1}{2i} \oint_C \overline{z}\, dz,$$

where A is the area of R. (Hint: Green's theorem)

5. In our proof of Cauchy's theorem, we implicitly assumed that the all the partial derivatives $\partial u/\partial x, \ldots, \partial v/\partial y$ were continuous when we invoked Green's theorem. Read worked problems 13–16 on pages 105 and 106 in the textbook where this extra assumption is removed. You do not need to submit any answer for this problem.