Math 113, Fall 2001

Solutions to Problem Set 7

December 17, 2001

1a. Suppose none of z_1, z_2, z_3, z_4 is equal to ∞. The cross-ratio (z_1, z_2, z_3, z_4) is equal to $(\frac{z_4-z_2}{z_4-z_1})\,(\frac{z_3-z_1}{z_3-z_2})$. Label the points z_1, z_2, z_3, z_4 in the complex plane by A, B, C, D respectively. Then the argument of the cross-ratio is simply the sum of the angles BDA and BCA, up to a sign. The sum of the angles BDA and BCA is 0 or π if and only if the points A, B, C, D all lie on a circle or line, and (z_1, z_2, z_3, z_4) is real if and only if its argument is 0 or π, whence the proposition.

If z_4 is equal to ∞, then the cross-ratio is simply $(\frac{z_3-z_1}{z_3-z_2})$. This cross-ratio is real if and only if z_1, z_2, z_3 are collinear. z_4 lies on all lines, so the cross-ratio is real if and only if all four points are collinear. Similar results hold if one of the other points is ∞.

1b. From Theorem 13.23 in Bak and Newman, the desired mapping ω is given by $\frac{(z-i)(-2)}{(z-1)(-1-i)} = \frac{(\omega+1)(1-i)}{(\omega-i)(2)}$. After some algebra, we get $\omega(z) = \frac{(1+2i)z+1}{z+(1-2i)}$.

2a. The function $f(z) = \frac{z-i}{z+i}$ maps $\mathbf{H}$ to the unit disc. The function $g(z) = iz$ maps the right half-plane to $\mathbf{H}$. Hence the function $f \circ g(z) = \frac{z-1}{z+1}$ is the function we seek.

2b. The function $h(z) = 1 - z$ maps the right half-plane to the half-plane $\{\mathrm{Re}(z) < 1\}$. Hence the function $f \circ g \circ h(z) = \frac{z}{z-2}$ does the job.

2c. The function $j(z) = e^{\frac{\pi z}{2}}$ maps the strip $\{-1 < \mathrm{Im}(z) < 1\}$ to the right half-plane, so the function we want is $f \circ g \circ j(z) = \frac{e^{\frac{\pi z}{2}}-1}{e^{\frac{\pi z}{2}}+1} = \tanh(\frac{\pi z}{4})$.

2d. The function $g(z) = iz$ maps the vertical strip $\{-1 < \mathrm{Re}(z) < 1\}$ to the horizontal strip $\{-1 < \mathrm{Re}(z) < 1\}$, so the function we want is $f \circ g \circ j \circ g(z) = \frac{e^{\frac{i\pi z}{2}}-1}{e^{\frac{i\pi z}{2}}+1} = i\tan(\frac{\pi z}{4})$.

3. Put $k(z) = \frac{z}{z-2}$, the function from **2b.** Note that $k(0) = 0$. Consider the function $k \circ f(z)$. This function maps Δ to itself analytically and sends 0 to 0. By Schwarz's lemma, $|(k \circ f)'(0)| \le 1$, or $|k'(f(0))f'(0)| \le 1$. One computes $k'(f(0))$ to be $-\frac{1}{2}$. Hence $|f'(0)| \le 2$. Also by Schwarz's lemma, $|k \circ f(z)| \le |z|$ for all $z \in \Delta$. Hence $|\frac{f(z)}{f(z)-2}| \le |z|$. Thus $|f(z)| \le |z||f(z)-2| \le |z||f(z)| + 2|z|$, so $|f(z)|(1-|z|) \le 2|z|$. Since $1 - |z|$ is positive for $z \in \Delta$, we have $|f(z)| \le \frac{2|z|}{1-|z|}$.

4. Let f be a conformal automorphism of Δ^*. Then f is bounded in a deleted neighborhood of 0. By Corollary 9.4 in Bak and Newman, f can be extended to an analytic function g on all of Δ. By the maximum principle, $|g(0)| < 1$. If $g(0) \in \Delta^*$, then there exists some $z \in \Delta^*$ such that $g(0) = g(z)$. Let U and V be disjoint neighborhoods of 0 and z, respectively. Since g is analytic, $g(U) \cap g(V)$ is open. But since g is injective on Δ^*, the only point in the intersection of $g(U) \cap g(V)$ is the point $g(0) = g(z)$, and this one-point is clearly not open, a contradiction. Hence we must have $g(0) = 0$. Thus g is an automorphism of the unit disc with $g(0) = 0$. By Lemma 13.14 in Bak and Newman, g is a rotation.

5. Fix $z_0 \in \Delta$. Put $g(z) = \frac{z+z_0}{1+\overline{z_0}z}$ and $h(z) = \frac{z-f(z_0)}{-\overline{f(z_0)}z+1}$. (Note that g and h depend on our choice of z_0.) Then $g(0) = z_0$, $h(z_0) = 0$ and $h \circ f \circ g(0)$ is a conformal automorphism of Δ fixing 0. By Schwarz's lemma, $|(h \circ f \circ g)'(0)| \leq 1$, or $|h'(f \circ g(0))f'(g(0))g'(0)| \leq 1$. One computes $g'(0)$ to be $(1 - |z_0|^2)$ and $h'(f \circ g(0))$ to be $\frac{1}{1-|f(z_0)|^2}$. Thus $f'(z_0) \leq \frac{1-|f(z_0)|^2}{1-|z_0|^2}$. This statement holds for any z_0 in Δ. Equality holds in Schwarz's lemma if and only if $h \circ f \circ g$ is a conformal automorphism, which happens if and only if f is a conformal automorphism.

6ab. If f is a conformal automorphism of Δ, then $\int_{f(\gamma)} \frac{|dz|}{1-|z|^2} = \int_\gamma \frac{|f'(z)||dz|}{1-|f(z)|^2}$. By Pick's lemma, $\frac{f'(z)}{1-|f(z)|^2} \leq 1 - |z|^2$, with equality holding if and only if f is an automorphism. Hence $\int_\gamma \frac{|f'(z)||dz|}{1-|f(z)|^2} \leq \int_\gamma \frac{|dz|}{1-|z|^2}$, with equality holding if and only if f is an automorphism.