Math 113, Fall 2001

Solutions to Problem Set 5

October 26, 2001

1. By computing derivatives, or by using various other tricks, we have the following power series expansions:

$$\frac{1}{(z+1)^2} = \sum_{n=0}^{\infty} \frac{(-1)^n n}{(2+i)^n}(z-(1+i))^n, \text{radius of convergence} = \sqrt{5}$$

$$e^z = \sum_{n=0}^{\infty} \frac{e^{(}1+i)}{n!}(z-(1+i))^n, \text{RoC} = \infty$$

$$z^3 = (1+i)^3 + 3(1+i)^2(z-(1+i)) + 3(1+i)(z-(1+i))^2 + (z-(1+i))^3, \text{RoC} = \infty$$

2a. Since $\sin z = \frac{e^{iz}-e^{-iz}}{2}$, $\sin z = 0$ if and only if $e^{iz} = e^{-iz}$, or $e^{2iz} = 1$. Hence $2iz = 2ni\pi$ for some integer n, so $z = n\pi$ for some integer n. Similarly one shows that $\cos z = 0$ if and only if $z = \frac{2n+1}{2}\pi$ for some integer n.

2b. The set of points $\{\frac{\pi}{3}, \frac{\pi}{4}, \dots\}$ has an accumulation point at 0. Since $0 \in U$, we must conclude that $f(z) = \tan z$ for all $z \in U$. Now $\cos z$ has zeroes at $\frac{2n+1}{2}\pi$ for all n, while $\sin z$ is nonzero at those points. Since $\tan z = \frac{\sin z}{\cos z}$, $\tan z$ cannot be extended to an analytic function at $z = \frac{2n+1}{2}\pi$. Thus $f(z)$ is not an entire function.

Note that $\tan z = \frac{1}{i}\frac{e^{iz}-e^{-iz}}{e^{iz}+e^{-iz}} = -i(1 - \frac{2}{e^{2iz}+1})$. If $\tan z = i$, then $e^{2iz} = 0$, which is impossible. Hence $\tan z \neq i$ for all z.

3a. First, we note that

$$\sin\frac{z}{2} = \frac{z}{2} - \frac{z^3}{2^3 3!} + \frac{z^5}{2^5 5!} - \dots$$

so that

$$\frac{2}{z}\sin\frac{z}{2} = 1 - \frac{z^2}{2^2 3!} + \frac{z^4}{2^4 5!} - \dots$$

Hence $\frac{2}{z}\sin\frac{z}{2} = 1$ at $z = 0$, so its reciprocal $\frac{\frac{z}{2}}{\sin\frac{z}{2}}$ admits an analytic extension to $z = 0$.

One easily checks that $f(z) = \frac{z}{2} \cot \frac{z}{2}$ is an even function; that is, $f(-z) = f(z)$. From this, one sees that the odd derivatives of f all vanish, so that all of the odd terms in the power series for $\frac{z}{2} \cot \frac{z}{2}$ all vanish.

3b. We note that $\cot \frac{z}{2} = i \, \frac{e^{\frac{iz}{2}} + e^{-\frac{iz}{2}}}{e^{\frac{iz}{2}} - e^{\frac{-iz}{2}}} = i \frac{e^{iz}+1}{e^{iz}-1} = i(1 + \frac{2}{e^{iz}-1})$. Hence $\frac{z}{2} \cot \frac{z}{2} = \frac{iz}{2} + \frac{iz}{e^{iz}-1}$.

We have that

$$e^{iz} - 1 = iz + \frac{(iz)^2}{2!} + \frac{(iz)^3}{3!} + \ldots,$$

so that

$$\frac{e^{iz} - 1}{iz} = 1 + \frac{iz}{2!} + \frac{(iz)^2}{3!} + \ldots.$$

Inverting this power series, we get

$$\frac{iz}{e^{iz} - 1} = 1 - \frac{iz}{2} + \frac{(iz)^2}{6 \cdot 2!} - \frac{(iz)^4}{30 \cdot 4!} + \frac{(iz)^6}{42 \cdot 6!} - \frac{(iz)^8}{30 \cdot 8!} + \ldots$$

Hence $B_2 = \frac{1}{6}$, $B_4 = \frac{1}{30}$, $B_6 = \frac{1}{42}$, $B_8 = \frac{1}{30}$. Note that some number theory texts use a different sign convention for Bernoulli numbers; they invert B_{4n} for all n, not that it matters for our purposes.

3c. The poles of $\frac{z}{2} \cot \frac{z}{2}$ that are closest to the origin are at $z = \pm 2\pi$, so the radius of convergence is 2π.

4a. Since $|e^z| = e^x$, the maximum and minimum moduli of e^z on D occur when x is maximized and minimized, and this happens on the boundary of D.

4b. Suppose $f(z)$ is a nonconstant polynomial with no zeroes. We have that $f(0) \neq 0$. For any disk centered at 0, the minimum modulus of $f(z)$ over the disk must occur at the boundary. But we have proven in a previous problem set that $f(z) \to \infty$ as $|z| \to \infty$, i.e. for all $N > 0$ there exists $R > 0$ such that $|f(z)| > N$ for all z such that $|z| > R$. Hence for a sufficiently large disk, all the boundary values have modulus greater than $|f(0)|$, a contradiction. Hence $f(z)$ must have a zero.

5a. Fix $z_0 \in E$. Regarding $F(z_0, w)$ as an analytic function of w, we have that $F(z_0, w)$ is 0 whenever $w \in E$. By the uniqueness principle, $F(z_0, w)$ must be 0 for all $w \in D$.

Now fix some $w_0 \in D$ (but not necessarily in E.) Regarding $F(z, w_0)$ as an analytic function of z, we have that $F(z, w_0)$ is 0 for all $z \in E$. Again by the uniqueness principle, $F(z, w_0)$ is 0 for all $z \in D$. Since w_0 is an arbitrary point of D, we have that $F(z, w) = 0$ for all $z, w \in D$.

5b. Since $e^{z+w} - e^z e^w = 0$ whenever z and w are in $\mathbf{R}$, and since $\mathbf{R}$ has accumulation points in $\mathbf{C}$, we conclude that $e^{z+w} - e^z e^w = 0$ for all $z, w \in \mathbf{C}$. Similarly $\cos\,(z + w) = \cos\, z \cos\, w$ - $\sin\, z \sin\, w$ for all $z, w \in \mathbf{C}$.

6a. Since f is an entire function, we can expand it in a power series about a. Write down

$$f(z) = f(a)+f'(a)(z-a)+\frac{f''(a)}{2!}(z-a)^2+\cdots+\frac{f^{(k)}(a)}{k!}(z-a)^k+(z-a)^{k+1}g(z).$$

We are simply factoring out the $(z-a)^{k+1}$ factor from the tail end of the power series, so $g(z)$ is an entire function. We thus have

$$\frac{f(z)}{(z-a)^{k+1}} = \frac{f(a)}{(z-a)^{k+1}} + \frac{f'(a)}{(z-a)^k} + \frac{f''(a)}{2!(z-a)^{k-1}} + \cdots + \frac{f^{(k)}(a)}{k!(z-a)} + g(z).$$

Integrating around C, we see that all the terms vanish except for the integral of $\frac{f^{(k)}(a)}{k!(z-a)}$, which equals $\frac{2\pi i f^{(k)}(a)}{k!}$. Hence we have

$$f^{(k)}(a) = \frac{k!}{2\pi i}\int_C \frac{f(z)}{(z-a)^{k+1}}dz,$$

as desired.

6b. Clearly, for m negative or 0,

$$\int_{|z|=1} \frac{e^{2z}}{z^m}dz = 0.$$

For $m \geq 1$,

$$\int_{|z|=1} \frac{e^{2z}}{z^m}dz = \frac{2\pi i}{(m-1)!}(\frac{d^{m-1}}{dz^{m-1}}e^{2z})(0) = \frac{2^m \pi i}{(m-1)!}.$$

6c. Using the ML-inequality on part (a), we have

$$|f^{(k)}(a)| \leq \frac{k!}{2\pi}\frac{M}{R^{k+1}}(2\pi R) = \frac{k!}{R^k}M.$$

Suppose $|f(z)| \leq M$ for all z. Then for all $a \in \mathbf{C}$ and all $R > 0$, we have

$$|f'(a)| \leq \frac{M}{R}.$$

Since this inequality holds for all R, we are forced to conclude that $f'(a) = 0$ for all a, so f is constant.