Math 113, Fall 2001

Solutions to Problem Set 3

October 15, 2001

1a. Write $e^z = e^{x+iy} = e^x e^{iy}$. Since $e^{iy} = \cos\, y + i\, \sin\, y$, one sees that for any real y_0, e^{iy} maps the interval $[y_0, y_0 + 2\pi)$ bijectively onto the unit circle. Now e^x maps the real line bijectively onto the set of positive real numbers. In addition, any complex number w not equal to zero has a unique representation as a product rv where r is real and v is on the unit circle; r is simply the magnitude of w. Hence the map e^z maps the strip of complex numbers A_{y_0} bijectively onto $\mathrm{C} - 0$.

1b. Clearly $\log_{y_0} e^{iy_0} = y_0$. Now if y_i is a sequence of real numbers approaching $y_0 + 2\pi$ from below, then clearly $\lim_{i\to\infty} \log_{y_0}(e^{iy_i}) = y_0 + 2\pi$. However, $\lim_{i\to\infty} e^{iy_i} = e^{iy_0}$, so $\log_{y_0} \lim_{i\to\infty} e^{iy_i} = y_0$. We have shown that the function $\log_{y_0}$ does not preserve limits of sequences, so it is not continuous.

1c. We see that $\log_{y_0} z_1 z_2 = \log|z_1 z_2| + i\theta$, where θ is some real number such that $e^{i\theta} = \frac{z_1 z_2}{|z_1 z_2|}$. In a similar manner we can write $\log_{y_0} z_1 = \log|z_1| + i\theta_1$ and $\log_{y_0} z_2 = \log|z_2| + i\theta_2$. Hence $\log_{y_0} z_1 + \log_{y_0} z_2 = \log|z_1| + \log|z_2| + i(\theta_1 + \theta_2)$. Clearly $\log|z_1| + \log|z_2| = \log|z_1 z_2|$. Since $e^{i\theta_1} = \frac{z_1}{|z_1|}$ and $e^{i\theta_2} = e^{i\theta_2} = \frac{z_2}{|z_2|}$, we see that $e^{i(\theta_1+\theta_2)} = \frac{z_1 z_2}{|z_1||z_2|} = \frac{z_1 z_2}{|z_1 z_2|} = e^{i\theta}$. Hence θ and $\theta_1 + \theta_2$ differ by a multiple of 2π. Thus $\log_{y_0} z_1 z_2$ and $\log_{y_0} z_1 + \log_{y_0} z_2$ differ by a multiple of $2\pi i$.

2. Let z_1 and z_2 be two points in the complex plane and let C be the straight-line segment joining them. By the complex-analytic version of the Fundamental Theorem of Calculus, $\int_C F' dz = F(z_2) - F(z_1)$. Of course, using the definition of line integrals and the fact that F' is identically zero, we deduce that $\int_C F' dz = 0$. Thus $F(z_2) = F(z_1)$. Since z_2 and z_1 were arbitrary, F must be constant over the whole plane.

Alternate method: Since F is analytic everywhere, $\frac{\partial F}{\partial \bar{z}} = 0$ everywhere. But since F' is zero everywhere, $\frac{\partial F}{\partial z} = 0$ everywhere. (See Homework 2, Problem 4 for the definitions of $\frac{\partial}{\partial z}$ and $\frac{\partial}{\partial \bar{z}}$.) Putting these two statements together, we can conclude that $\frac{\partial F}{\partial x}$ and $\frac{\partial F}{\partial y}$ are 0 everywhere, so that F is constant.

3. Let C_1 be the segment joining 0 to 1, C_2 the segment joining 1 to $1 + i$, C_3 the segment joining $1 + i$ to i, and C_4 the segment joining i to 0.

Parametrize C_1 as follows: define $z : [0, 1] \to \mathrm{C}$ by setting $z(t) = t$. Then $\int_{C_1} x\, dz = \int_0^1 t\, dt = \frac{1}{2}$.

Parametrize C_2 by setting $z(t) = 1 + it$. Then $\int_{C_2} x\ dz = \int_0^1 1\ (i\ dt) = i$.

Parametrize C_3 by setting $z(t) = (1-t) + i$. Then $\int_{C_3} x\ dz = \int_0^1 (1-t)\ (-dt) = -\frac{1}{2}$.

Parametrize C_4 by setting $z(t) = i(1-t)$. Then $\int_{C_4} x\ dz = \int_0^1 0\ (-i\ dt) = 0$.

$\int_C x\ dz$ is the sum of these four integrals, which is i.

4a. Using the given parametrization of C, we have that $\int_C (z-i)dz = \int_{-1}^1 (t + i(t^2-1))(1+2it)dt = \int_{-1}^1 (3t - 2t^3) + i(3t^2-1)dt$. This last integral is equal to 0.

4b. The function $F(z) = \frac{z^2}{2} - iz$ is an antiderivative for $z - i$. The endpoints of C are $1+i$ and $-1+i$. Hence $\int_C (z-i)dz = F(1+i) - F(-1+i) = 1 - 1 = 0$.

4c. If D is the straight line segment joining $1+i$ and $-1+i$, parametrize D by defining $z(t) = -t + i$. Here t runs from -1 to 1. Then $\int_D (z-i)dz = \int_0^1 -t\ dt$ = 0. Since C followed by D forms a closed curve, the sum of the integral over C and the integral over D must equal 0. Since the integral over D equals 0, the integral over C is 0.

5. Note that along the upper half of the unit circle, $|z|$ is always 1. Since $e^z = e^x e^{iy}$, $|e^z| = |e^x|$. Thus the maximum value of $|e^z|$ over the unit circle is $e^1 = e$. Hence the magnitude of the integrand $\frac{e^z}{z}$ is bounded above by e. Since the perimeter of the upper half of the unit circle is π, the ML-inequality tells us that

$$|\int_C \frac{e^z}{z} dz| \leq \pi e.$$

6. Using the hint in Bak and Newman: Put $\int_{S^1} f(z)dz = Re^{i\theta}$, where R and θ are real. Then $|\int_{S^1} f(z)dz| = |\int_{S^1} f(z)e^{-i\theta}dz|$. This second integral is real, so we can equate $\int_{S^1} f(z)e^{-i\theta}dz$ with its real part. Using the parametrization $z = e^{it}$ for t between 0 and 2π, we get that $\int_{S^1} f(z)e^{-i\theta}dz$ equals $\int_0^{2\pi} f(e^{it})e^{-i\theta}ie^{it}dt$. Hence the real part of this integral is equal to $\int_0^{2\pi} f(e^{it})(-\sin(t-\theta))dt$. Now $|\int_0^{2\pi} f(e^{it})(-\sin(t-\theta))dt| \leq \int_0^{2\pi} |f(e^{it})(-\sin(t-\theta))|dt \leq \int_0^{2\pi} M|\sin(t-\theta)| = 4M$. Hence $|\int_{S^1} f(z)dz| \leq 4M$.