Midterm Exam

Due Thursday, November 1

There are a total of 75 possible points on this test.

You may not collaborate with anyone else on this exam. You may use the class textbook, your class notes, your homework write-ups, and the homework solutions handed out by the CA's as references on this exam. No other resources are allowed – in particular, you may not use any books other than the class textbook. By taking this exam, you are agreeing to obey by these rules. Any violation of these rules, if detected, will result in a score of zero on the exam.

You may refer to previous homework problems and any theorems we have proved in class in your solutions to the following problems.

Please type or write neatly. Try to be both concise and rigorous whenever possible. Good luck!!

1. (10 points)

   a. Compute all the complex cube roots of $8i$.

   b. Fix an integer $n \geq 1$, and let $\omega_0, \omega_1, \ldots, \omega_{n-1}$ be the nth roots of unity. Let k be an integer such that $1 \leq k \leq n$. Show that

$$\sum_{j=0}^{n-1} \omega_j^k = \begin{cases} 0, & 1 \leq k \leq n-1, \\ n, & k = n. \end{cases}$$

2. (15 points) Define the *hyperbolic cosine* and *hyperbolic sine* for $z \in \mathbf{C}$ by the rules

$$\cosh(z) = \frac{e^z + e^{-z}}{2}, \ \sinh(z) = \frac{e^z - e^{-z}}{2}.$$

   a. Find all zeros of $\cosh(z)$, and plot them in the complex plane.

   b. Let $\tanh(z) = \sinh(z)/\cosh(z)$ whenever the right-hand side is defined. Evaluate the contour integral

$$\int_{|z-2|=1} \tanh(z) dz.$$

c. Prove that the function $f : \mathbf{C} \to \mathbf{C}$ defined by $f(z) = \cosh^2(z) + \cosh(z)$ is surjective.

3. (10 points) Let C be the path in $\mathbf{C}$ which goes from 1 to $1+i$ along a straight line. Evaluate the contour integral

$$\int_C \left(\overline{z}^2 + \frac{1}{z} \right) dz.$$

4. (10 points) If f is analytic on and inside the unit disk, show that for $0 \leq t < 2\pi$, we have

$$f(re^{it}) = \frac{1}{2\pi} \int_0^{2\pi} \frac{f(e^{i\theta})}{1 - re^{i(t-\theta)}} d\theta$$

for all real numbers r with $0 \leq r < 1$.

5. (10 points) Let D be the open unit disk $|z| < 1$. Suppose $f : D \to \mathbf{C}$ is analytic on D and that $|f^{(k)}(0)| \leq 2^k$ for all positive integers k. Prove that f can be extended to an entire function, i.e., that there exists an entire function $g : \mathbf{C} \to \mathbf{C}$ such that $f(z) = g(z)$ for all $z \in D$.

6. (10 points) Let f be an entire function, and suppose that there exists a real number $M > 0$ such that

$$|f(z + \frac{1}{n})| \leq M|f(z)|$$

for all $z \in \mathbf{C}$ and all positive integers n. Prove that there exists a complex number a such that $f(z+1) = af(z)$ for all $z \in \mathbf{C}$.

7. (10 points) Let $c_0, c_1, \ldots, c_n$ be complex numbers, and assume that $c_k \neq 0$ for some integer $1 \leq k \leq n$. Define a function $f : \mathbf{R} \to \mathbf{C}$ by

$$f(\theta) = c_0 + c_1 e^{i\theta} + c_2 e^{2i\theta} + \cdots + c_n e^{ni\theta}.$$

Prove that there exists some θ_0 with $0 \leq \theta_0 < 2\pi$ such that $|f(\theta_0)| > |c_0|$. Your solution should *not* involve any complicated estimates or computations...