# Homework Assignment # 9

Due Thursday, December 13

Throughout this assignment, $\Delta$ will denote the open unit disk $\{|z| < 1\}$.

1. a. Prove that the cross-ratio $(z_1, z_2, z_3, z_4)$ is real-valued iff the four points $z_1, z_2, z_3, z_4$ all lie on a common circle or line.
   b. Find the unique bilinear mapping sending $1, i, -1$ onto $i, -1, 1$ respectively.

2. Find explicit conformal mappings of the following domains onto $\Delta$:
   a. The right half-plane $\{\text{Re}(z) > 0\}$.
   b. The half-plane $\{\text{Re}(z) < 1\}$. [**Hint:** Use part (a).]
   c. The horizontal strip $\{-1 < \text{Im}(z) < 1\}$. [**Hint:** Use part (a) and the exponential map.]
   d. The vertical strip $\{-1 < \text{Re}(z) < 1\}$. [**Hint:** Use part (c).]

3. Suppose $f$ is an analytic function on $\Delta$ with $f(0) = 0$ and $\text{Re}(f(z)) < 1$ for all $z \in \Delta$. Show that $|f'(0)| \leq 2$ and that $|f(z)| \leq \frac{2|z|}{1-|z|}$ for all $z \in \Delta$. [**Hint:** Consider the composition of $f$ and the mapping from Exercise 2, part (b).]

4. Prove that every conformal automorphism of the punctured disk $\Delta^* = \{0 < |z| < 1\}$ is a rotation. [**Hint:** Use Riemann's removable singularity theorem.]

5. Prove *Pick's lemma*: If $f : \Delta \to \Delta$ is analytic, then

$$|f'(z)| \leq \frac{1 - |f(z)|^2}{1 - |z|^2}$$

for all $z \in \Delta$, with equality iff $f$ is a conformal automorphism of $\Delta$. [**Hint:** Fix $z_0 \in \mathbf{C}$, let $g(z)$ be an automorphism of $\Delta$ taking 0 to $z_0$, let $h(z)$ be an automorphism of $\Delta$ taking $f(z_0)$ to 0, and apply the Schwarz lemma to $h \circ f \circ g$.]

6. Let $D \subseteq \mathbf{C}$ be a domain, and let $\gamma \subseteq D$ be a simple smooth curve, parametrized by $z(t)$ for $a \leq t \leq b$. Let $f : D \to \mathbf{C}$ be a $C^1$ function (where $C^1$ means continuously differentiable as a map from $D \subset \mathbf{R}^2$ to $\mathbf{R}^2$). We define the integral $\int_\gamma f(z)|dz|$ to be

$$\int_\gamma f(z)|dz| = \int_a^b f(z(t))|\dot{z}(t)|dt.$$

The length of $\gamma$ in the Euclidean metric is, by definition, the integral $\int_\gamma |dz|$.

On the other hand, if $\gamma$ is a smooth curve contained in $\Delta$, we define the length of $\gamma$ in the *hyperbolic metric* to be

$$\int_\gamma \frac{|dz|}{1 - |z|^2}.$$

a. Prove that if $f$ is a conformal automorphism of $\Delta$, then $\gamma$ and $f(\gamma)$ have the same length in the hyperbolic metric. (One can show that this is false for the Euclidean metric.) [**Hint:** Use Pick's lemma.]

b. If $f : \Delta \to \Delta$ is an analytic map which is *not* a bijection, show that $f(\gamma)$ is *shorter* than $\gamma$ in the hyperbolic metric.