Homework Assignment # 7

Due Thursday, November 29

This assignment will count as **two** homework assignments, and you will have two weeks to do it.

1. Find all residues at all singularities of:
   a. $z\cot(z)$
   b. $\frac{z-1}{(z^4-1)^2}$
   c. $\sin(1/z)$
   d. $\frac{1}{e^z-1}$

2. Evaluate
$$\int_{|z|=2} (2z-1)e^{(z-1)/z}dz.$$

3. a. Let A, B be analytic at z_0 and suppose that $A(z)$ has a zero of order k at z_0, and that $B(z)$ has a zero of order $k+1$ at z_0. Prove that $A(z)/B(z)$ has a simple pole at z_0, and that
$$\text{Res}(\frac{A(z)}{B(z)}; z_0) = (k+1)\frac{A^{(k)}(z_0)}{B^{(k+1)}(z_0)}.$$
   Use this to evaluate
$$\int_{|z|=1} \frac{z^3}{(1-\cos(z))^2}dz.$$

   b. Let A, B be analytic at z_0, with $A(z_0) \neq 0$ and $B(z)$ having a zero of order 2 at z_0. Prove that $A(z)/B(z)$ has a double pole at z_0, and that
$$\text{Res}(\frac{A(z)}{B(z)}; z_0) = 2\frac{A'(z_0)}{B''(z_0)} - \frac{2A(z_0)B'''(z_0)}{3(B''(z_0))^2}.$$
   Use this to evaluate
$$\int_{|z-1|=1} \frac{e^z}{(z-1)^2}dz.$$

4. Evaluate the following definite integrals:

   a. $\int_{-\infty}^{\infty} \frac{x^2 dx}{1+x^4}$

   b. $\int_0^{2\pi} \frac{d\theta}{1+\sin^2\theta}$

   c. $\int_0^{\infty} \frac{dx}{1+x^n}$, $n \in \mathbf{Z}, n \geq 2$. (See p.147 of the book for a good contour to use.)

5. a. For any real number $a > 0$, show that

$$\sum_{n=0}^{\infty} \frac{1}{n^2 + a^2} = \frac{\pi}{2a} \coth(\pi a) + \frac{1}{2a^2},$$

   where coth means hyperbolic cotangent.

   b. Evaluate

$$\sum_{n=0}^{\infty} \binom{3n}{n} \frac{1}{8^n}.$$

6. Prove the "Fractional Residue Theorem": Suppose f is analytic with a simple pole at z_0, and let γ_r be a circular arc with an angle of α radians on a circle of radius r centered at z_0. Show that

$$\lim_{r \to 0} \int_{\gamma_r} f = \alpha i \mathrm{Res}(f; z_0).$$

7. Suppose $f(z)$ has an isolated singularity at ∞. We define the *residue of f at infinity* to be

$$\mathrm{Res}(f(z); \infty) := \mathrm{Res}(-\frac{1}{w^2} f(1/w); w = 0).$$

   [**Aside:** The reason for this strange definition, if you're curious, is that when working with manifolds (like the Riemann sphere), it does not make sense to take the residue of a function at point; the notion of residue only behaves well under changes of coordinates for differential forms. So the residue of $f(z)$ at a point z_0 is really the residue of the differential form $f(z)dz$ at z_0. And if we make the change of variables $w = 1/z$, then $f(z)dz$ is transformed into $f(1/w)d(1/w) = -\frac{1}{w^2} f(1/w)dw$. Even if you don't know about differential forms, this should give you a good mnemonic for remembering the definition of the residue at infinity.]

a. One has to be a bit careful with residues at infinity: Show by example that even if f has a removable singularity (i.e., is analytic) at ∞, we might have $\text{Res}(f(z);\infty) \neq 0$.

[**Remark:** In terms of differential forms, the interpretation of this seemingly puzzling fact is that although $f(z)$ is analytic at ∞, the 1-form dz (and therefore also $f(z)dz$) has a double pole at infinity.]

b. Suppose f is analytic on the set $\{|z| > R_0\}$, and let $R > R_0$. Prove that

$$\int_{|z|=R} f(z)dz = -2\pi i \text{Res}(f;\infty).$$

By considering the Riemann sphere, explain intuitively why one gets a minus sign in this formula.

c. Suppose that f is analytic on $\mathbf{C}$ except at a finite number of isolated singularities. Prove that

$$\sum_{z_0 \in \hat{\mathbf{C}}} \text{Res}(f(z); z_0) = 0,$$

where the sum runs over all points of the Riemann sphere $\hat{\mathbf{C}}$ (i.e., $\mathbf{C}$ plus ∞). [**Hint:** Consider the line integral of f over a large circle and use part (b).]

d. If γ is a regular closed curve in $\mathbf{C}$ and f is analytic along γ, with only finitely many isolated singularities outside γ, show that

$$\int_\gamma f(z)dz = -2\pi i \sum \{\text{residues of } f \text{ outside } \gamma \text{ including } \infty\}.$$

e. If $P(z)$ and $Q(z)$ are polynomials and $\deg(Q) - \deg(P) \geq 2$, prove that the residue of $f(z) = \frac{P(z)}{Q(z)}$ at infinity is zero.

f. Evaluate the contour integral

$$\int_{|z|=1} \frac{dz}{(z-2)(1+2z)^4(1-3z)^7}.$$

[**Hint:** Use parts (d) and (e). Notice that the integrand has high-order poles inside the unit circle but low-order poles outside the unit circle, so using the residue at infinity is *much* simpler than doing this integral the "usual" way!]