Homework Assignment # 6

Due Thursday, November 15

1. a. If Log denotes the principal branch of the logarithm, simplify the expressions
$$e^{\text{Log}(i)}, \text{Log}(-i), i^{\text{Log}(-1)}, (1+i)^{\text{Log}(1+i)},$$
where $z^w = e^{w\text{Log}z}$.

   b. Find the Taylor expansion of Log(z) around $z = 1$.

2. Show that
$$f(z) = \int_0^1 \frac{\sin(zt)}{t} dt$$
is an entire function (a) by applying Morera's theorem, and (b) by obtaining a power series expansion for f. What is $f'(z)$?

3. Find Laurent series expansions for the following functions around $z_0 = 0$, and indicate the regions in which they are valid:

   a. $\sin(1/z)$

   b. $z/(z+1)$

   c. e^z/z^2

   d. $1/(e^z - 1)$ (Find the first three terms only)

   e. $\frac{(z-1)^2(z+3)}{1-\sin(\pi z/2)}$ (Find the first three terms only).

   Classify the type of singularity at zero, if there is one, in each of the above five examples.

4. Prove the following complex version of l'Hôpital's rule: Let $f(z), g(z)$ be analytic, both having zeros of order k at z_0. Then $f(z)/g(z)$ has a removable singularity at z_0, and
$$\lim_{z\to z_0} \frac{f(z)}{g(z)} = \frac{f^{(k)}(z_0)}{g^{(k)}(z_0)}.$$

5. We say that a function $f(z)$ has an *isolated singularity at infinity* iff $g(w) = f(1/w)$ has an isolated singularity at $w = 0$.

a. Show that f has an isolated singularity at infinity iff f is analytic outside of some bounded subset of $\mathbf{C}$. Sketch a picture of what this means in terms of the Riemann sphere.

b. Show that a polynomial of degree $N \geq 1$ has a pole of order N at infinity.

c. Show that an entire function which is not a polynomial must have an essential singularity at ∞.

6. Suppose $f(z)$ is analytic in the annulus $R_1 < |z| < R_2$, and let r be such that $R_1 < r < R_2$.

   a. Show that f can be developed on the circle of radius r around $z = 0$ as a *Fourier series*:

   $$f(re^{i\theta}) = \sum_{n=-\infty}^{\infty} b_n e^{in\theta}.$$

   b. Show that the Fourier coefficients b_n can be calculated directly from f using the Fourier inversion formula

   $$b_n = \frac{1}{2\pi} \int_0^{2\pi} f(re^{i\theta}) e^{-in\theta} d\theta.$$

   c. Prove Parseval's formula:

   $$\frac{1}{2\pi} \int_0^{2\pi} |f(re^{i\theta})|^2 d\theta = \sum_{n=-\infty}^{\infty} |b_n|^2.$$