

HOMEWORK ASSIGNMENT # 5

Due Thursday, October 25

1. Find power series expansions for $f(z) = \frac{1}{(z+1)^2}$, $g(z) = e^z$, and $h(z) = z^3$ centered at $z = 1 + i$. What are the radii of convergence of these power series?
2. a. Find all complex zeros of the functions $\sin(z)$ and $\cos(z)$.
b. Suppose f is an analytic function on the open set $U = \mathbf{C} \setminus \{\frac{\pi}{2} \pm \pi k \mid k \in \mathbf{Z}\}$, and that $f(\frac{\pi}{n}) = \tan(\frac{\pi}{n})$ for all integers $n \geq 3$. Show that f is not entire, and that f does not attain the value i .
3. Define the *Bernoulli numbers* B_{2n} by the formula

$$\frac{z}{2} \cot\left(\frac{z}{2}\right) = 1 - B_2 \frac{z^2}{2!} - B_4 \frac{z^4}{4!} - B_6 \frac{z^6}{6!} - \dots$$

(This sequence of rational numbers is very important in number theory, and is intimately connected with both the distribution of prime numbers and Fermat's Last Theorem.)

- a. Explain why there are no odd terms in the above power series. [**Hint:** Prove that this is a general property of odd functions.]
 - b. Compute the Bernoulli numbers B_2, B_4, B_6, B_8 .
 - c. What is the radius of convergence of the above power series?
4. a. Show directly that the maximum and minimum moduli of e^z on a compact domain D are assumed on the boundary of D .
b. Use the Minimum Modulus Principle to give another proof of the Fundamental Theorem of Algebra.
 5. a. Let $D \subseteq \mathbf{C}$ be a domain, and let $E \subseteq D$ be a subset containing an accumulation point. Let $F(z, w) : D \times D \rightarrow \mathbf{C}$ be a function which is analytic in z for each fixed $w \in D$, and analytic in w for each fixed $z \in D$. If $F(z, w) = 0$ whenever $z, w \in E$, show that $F(z, w) = 0$ for all $z, w \in D$.

- b. Use part (a) to show that $e^{z+w} = e^z \cdot e^w$ and $\cos(z+w) = \cos(z)\cos(w) - \sin(z)\sin(w)$ for all $z, w \in \mathbf{C}$, assuming the corresponding identities for $z, w \in \mathbf{R}$. [Use the power series definitions for $e^z, \sin(z), \cos(z)$ as was done in class.]
6. a. Prove the following generalized Cauchy integral formula: If f is entire, C is a circle of radius R centered at z_0 , and if a is inside C , then

$$f^{(k)}(a) = \frac{k!}{2\pi i} \int_C \frac{f(z)}{(z-a)^{k+1}} dz.$$

[**Hint:** First find complex numbers $c_0, \dots, c_k \in \mathbf{C}$ such that

$$\int_C \frac{f(z)}{(z-a)^{k+1}} dz = c_0 \int_C \frac{dz}{(z-a)^{k+1}} + c_1 \int_C \frac{dz}{(z-a)^k} + \dots + c_k \int_C \frac{dz}{(z-a)} + \int_C g(z) dz$$

with g entire.]

- b. Use part (a) to evaluate

$$\int_{|z|=1} \frac{e^{2z}}{z^m} dz$$

for all $m \in \mathbf{Z}$.

- c. Let $z_0 \in \mathbf{C}$, and suppose $f(z)$ is entire. If $|f(z)| \leq M$ whenever $|z - z_0| = R$, show that

$$|f^{(k)}(z_0)| \leq \frac{k!}{R^k} M$$

for all $k \geq 0$. Use this inequality with $k = 1$ to give another proof of Liouville's theorem.