Homework Assignment # 4

Due Thursday, October 18

1. Let $f : \mathbf{C} \to \mathbf{C}$ be a continuous function, and let C be a smooth curve in $\mathbf{C}$ parametrized by $z(t) : [a, b] \to \mathbf{C}$. Let $P = (a_0 = a, a_1, a_2, \ldots, a_n = b)$ be a partition of $[a, b]$, let $z_j = z(a_j)$ for $j = 0, \ldots, n$, and let $R_P(f) = \sum_{j=1}^{n} f(z_j)(z_j - z_{j-1})$. Explain why $\int_C f \approx R_P(f)$ if $\max_j |a_j - a_{j-1}|$ is small. Formulate and prove a more precise version of this statement.

2. We know from lecture that if $f(z)$ is analytic on the closed disc $\{|z| \le R\}$ then $\int_{|z|=R} f(z)dz = 0$. Show that this conclusion remains true if we just assume that $f(z)$ is continuous on the closed disc $\{|z| \le R\}$ and analytic on the open disc $\{|z| < R\}$ [**Hint:** Approximate $f(z)$ uniformly with the functions $f_r(z) = f(rz)$ for $0 < r < 1$.]

3. Evaluate the contour integral
$$\int_{|z|=3} \frac{dz}{1+z^2}$$
by using the Cauchy integral formula and partial fractions.

4. A function $f(z)$ on $\mathbf{C}$ is said to be *doubly periodic* if there exist complex numbers ω_1, ω_2 (called the *periods* of f), linearly independent over $\mathbf{R}$, such that
$$f(z + \omega_1) = f(z + \omega_2) = f(z)$$
for all $z \in \mathbf{C}$. Show that a doubly periodic entire function must be constant.

5. Recall the following statement of Green's theorem (which is slightly more general than the version presented in lecture). Let Γ be a smooth, simple closed curve, oriented counterclockwise, which surrounds the region D in $\mathbf{R}^2$. Denote by $\overline{D}$ the closure of D, i.e., $\overline{D} = D \cup \Gamma$. Let $P, Q : \overline{D} \to \mathbf{R}^2$ be continuously differentiable (C^1) real-valued functions. Then
$$\int_\Gamma P dx + Q dy = \int \int_D (Q_x - P_y) dx dy.$$

a. Use Green's theorem to prove that if $g(z)$ is a C^1 function on $\overline{D} \subset \mathbf{C}$, then

$$\int_\Gamma g(z)dz = 2i \int\int_D \frac{\partial g}{\partial \overline{z}} dxdy.$$

(See homework 2 for the meaning of $\frac{\partial}{\partial \overline{z}}$). Explain why this formula implies the closed curve theorem if g is C^1 and entire.

b. Show that

$$\text{Area}(D) = \frac{1}{2i}\int_\Gamma \overline{z}dz = \int_\Gamma xdy.$$

c. Hiker Bob, walking on a sheet of graph paper, begins at the origin and then traces out a simple closed curve Γ, always walking along the grid (i.e., always moving an integral number of units in one of the four coodrinate directions). Bob moves a total of 6 units north along points with x-coordinate 3, 4 units north with x-coordinate 7, 2 units south with x-coordinate 2, and 8 units south with x-coordinate 0. Use part (b) to find the area of the region enclosed by Γ.

6. Let $f(z) = c_0 + c_1 z + \cdots + c_n z^n$ be a polynomial with $c_k \in \mathbf{C}$ for all k. Prove that

$$2\int_{-1}^{1} f(x)^2 dx \le \int_0^{2\pi} |f(e^{i\theta})|^2 d\theta = 2\pi \sum_{k=0}^{n} |c_k|^2.$$

[**Hint:** For the first inequality, apply the closed curve theorem to the function $f(z)^2$ on the top and bottom halves of the unit disc. For the second, first work out the value of $\int_0^{2\pi} e^{ik\theta} d\theta$ for $k \in \mathbf{Z}$.]