Final Examination Solutions

1. (15 points)

a. State the Cauchy-Riemann equations for the real and imaginary parts of an analytic function.

Solution:

Suppose $f = u + iv$ is analytic. Then it satisfies the Cauchy-Riemann equations $u_x = v_y, u_y = -v_x$. Equivalently, $f_y = if_x$.

b. Define the winding number of a closed curve γ around a point $a \notin \gamma$.

Solution:

The winding number is given by

$$\frac{1}{2\pi i}\int_\gamma \frac{dz}{z-a}.$$

c. State the Schwarz lemma (including the condition for equality).

Solution:

Let Δ be the open unit disk $|z| < 1$. Suppose $f : \Delta \to \Delta$ is analytic, and $f(0) = 0$. Then $|f(z)| \leq |z|$ for all $z \in \Delta$ and $|f'(0)| \leq 1$. We have equality in either case iff f is a rotation.

2. (10 points)

a. Determine and classify all singularities of

$$f(z) = z^3 e^{1/z} + \frac{z^2}{(z-1)^3}$$

and calculate the residues at each.

Solution:

$f(z)$ has an essential singularity at $z = 0$ and has a triple pole at $z = 1$. Note that the singularities of $z^3 e^{1/z}$ and $\frac{z^2}{(z-1)^3}$ are "independent", so that we can calculate the residues separately. Therefore the residue of $f(z)$ at $z = 0$ is the residue of $z^3 e^{1/z}$ at $z = 0$, which by the Laurent expansion $z^3(1 + 1/z + 1/(2!z^2) + \cdots)$ equals $1/24$. And the residue of $f(z)$ at $z = 1$ is the residue of $z^2/(z-1)^3 = 1/(z-1)^3 + 2/(z-1)^2 + 1/(z-1)$ at $z = 1$, which is 1.

b. Let $\tan(z) = \sum_{n=0}^{\infty} a_n z^n$ be the Taylor expansion of $\tan(z)$ around $z = 0$. Does the series

$$\sum_{n=1}^{\infty} \frac{3^n a_n}{2^n}$$

converge or diverge? Justify your answer.

Solution:

The series in question is the Taylor series for $\tan(z)$ around $z = 0$, evaluated at $z = 3/2$. We therefore need to calculate the radius of convergence R of this Taylor series, which we know is equal to the distance from $z = 0$ to the nearest singularity of $\tan(z)$, which occurs at $z = \pm\pi/2$. Therefore $R = \pi/2 > 3/2$, so the answer is that the original series *converges.*

3. (10 points) Let A be the complement in $\mathbf{C}$ of the non-positive real axis, i.e., $A = \mathbf{C} - \{x \in \mathbf{R} \mid x \leq 0\}$.

a. Show that there exists a unique function f which is analytic on A and satisfies $f(x) = x^x$ for all real $x > 0$.

Solution:

The function $f(z) = e^{z\mathrm{Log(z)}}$ clearly has the desired properties, where $\mathrm{Log}(z)$ denotes the principal branch of the logarithm. Uniqueness follows from the uniqueness principle for analytic functions, since A is connected.

b. Find $f(i)$ and $f'(i)$.

Solution:

We have

$$f(i) = e^{i(\log|i| + i\mathrm{Arg}(i))} = e^{i \cdot i\pi/2} = e^{-\pi/2}.$$

Similarly, as $f'(z) = f(z)(1 + \mathrm{Log}(z))$, we have

$$f'(i) = e^{-\pi/2}(1 + i\pi/2).$$

4. (10 points) Let $P(z) = a_0 + a_1 z + \cdots + a_k z^k$ be a complex polynomial. Suppose $|P(z)| \leq 1$ whenever $|z| \leq 1$. Show that $|a_n| \leq 1$ for all $n = 0, 1, \ldots, k$.

Solution:

By the residue theorem, we have

$$a_n = \frac{1}{2\pi i} \int_{|z|=1} \frac{P(z)}{z^{n+1}} dz,$$

so that

$$|a_n| \leq \frac{1}{2\pi} \cdot 1 \cdot 2\pi = 1$$

by the ML inequality.

Alternate solution:

By Parseval's identity, we have

$$\frac{1}{2\pi} \int_0^{2\pi} |P(e^{i\theta})|^2 d\theta = \sum_{n=0}^{k} |a_n|^2,$$

and by ML we thus have $\sum_{n=0}^{k} |a_n|^2 \leq 1$, which clearly implies that $|a_n| \leq 1$ for all n.

5. (15 points)

a. State Rouché's theorem.

Solution:

Let g, h be analytic functions on and inside the simple closed curve γ, and suppose that $|g| > |h|$ on γ. Then g and $g + h$ have the same number of zeros inside γ (counting multiplicities).

b. How many zeros (counting multiplicities) does the function $f(z) = e^z - 3z^2$ have inside the unit circle?

Solution:

Let $h(z) = e^z, g(z) = -3z^2$. On the unit circle, $|h(z)| \leq e < 3$ and $|g(z)| = 3$, so by Rouché's theorem, $f(z)$ and $g(z)$ have the same number of zeros (counting multiplicities) inside the unit circle, namely 2.

c. How many *distinct* zeros does f have inside the unit circle?

Solution:

Suppose z_0 is a zero of f inside the unit circle. Then $e^{z_0} = 3z_0^2$ and

$$f'(z_0) = e^{z_0} - 6z_0 = 3z_0^2 - 6z_0 = z_0(3z_0 - 6) \neq 0$$

since clearly $z_0 \neq 0, 2$. So all zeros of f are simple, and therefore f has 2 distinct zeros inside the unit circle.

Alternate solution:

By the intermediate value theorem, f has at least one real zero between -1 and 0 and another between 0 and 1, since $f(-1) < 0, f(0) > 0, f(1) < 0$. Since we know f has at most 2 distinct zeros inside the unit circle, it follows that it has exactly 2.

6. (15 points) Let $B = \{Re^{i\theta} \mid R > 0, -\pi/4 < \theta < \pi/4.\}$.

   a. Find an explicit analytic function ϕ mapping B onto the open unit disk Δ given by $|z| < 1$.

   Solution:

   The map $z \mapsto z^2$ takes B onto the right half-plane, $z \mapsto iz$ takes the right half-plane to the upper half-plane, and $z \mapsto \frac{z-i}{z+i}$ takes the upper half-plane to the unit disk. The composite of these maps,

$$\phi(z) := \frac{iz^2 - i}{iz^2 + i} = \frac{z^2 - 1}{z^2 + 1},$$

   is the desired function.

   b. Prove that there are no non-constant analytic maps $f : \mathbf{C} \to B$.

   Solution:

   If $f : \mathbf{C} \to B$ is analytic, the composite map $\phi \circ f$ is an analytic map from $\mathbf{C}$ to Δ, hence is a bounded entire function, which by Liouville's theorem must be constant. Since ϕ is injective, this implies that f is constant as well, as desired.

7. (10 points) Evaluate the integral

$$\int_0^{2\pi} \frac{d\theta}{3 + \cos\theta}.$$

Solution:

Making the substitution $z = e^{i\theta}$, we have $d\theta = \frac{dz}{iz}$ and $\cos\theta = \frac{1}{2}(z + \frac{1}{z})$, so that

$$\int_0^{2\pi} \frac{d\theta}{3 + \cos\theta} = \frac{2}{i} \int_{|z|=1} \frac{dz}{z^2 + 6z + 1}.$$

The roots of $z^2 + 6z + 1$ are $z_1 = -3 + 2\sqrt{2}$, which is inside the unit circle, and $z_2 = -3 - 2\sqrt{2}$, which is outside. The residue at $z = z_1$ of $1/(z^2+6z+1)$ is $1/(z_1 - z_2) = \frac{1}{4\sqrt{2}}$. By the residue theorem, we find that

$$\int_0^{2\pi} \frac{d\theta}{3+\cos\theta} = \frac{2}{i}(2\pi i)(\frac{1}{4\sqrt{2}}) = \frac{\pi}{\sqrt{2}}.$$

8. (15 points) Let $R > 1$ be a real number, and let C_R be the semicircle parametrized by $z(\theta) = Re^{i\theta}$, $0 \leq \theta \leq \pi$. Let $a > 0$ be a real number.

   a. If $z \in C_R$, show that

   $$|\frac{e^{iaz}}{1+z^2}| \leq \frac{1}{R^2-1}.$$

   Solution:
   If $z \in C_R$, then

   $$|\frac{e^{iaz}}{1+z^2}| \leq \frac{e^{\mathrm{Re}(iaz)}}{|1+z^2|} \leq \frac{e^{-a\mathrm{Im}(z)}}{|1+z^2|} \leq \frac{1}{|1+z^2|} \leq \frac{1}{R^2-1},$$

   where the last step is by the (reverse) triangle inequality.

   b. Evaluate

   $$\int_{-\infty}^{\infty} \frac{\cos(ax)}{1+x^2}\,dx.$$

   (**Hint:** Use part (a).)

   Solution:
   We have

   $$\int_{-\infty}^{\infty} \frac{\cos(ax)}{1+x^2}\,dx = \int_{-\infty}^{\infty} \mathrm{Re}\left(\frac{e^{iax}}{1+x^2}\right)\,dx = \mathrm{Re}\left(\int_{-\infty}^{\infty} \frac{e^{iax}}{1+x^2}\,dx\right).$$

   Since this integral converges absolutely by comparison with $\int_{-\infty}^{\infty} \frac{1}{1+x^2}\,dx$, we can compute it as

   $$\mathrm{Re}\left(\lim_{R\to\infty} \int_{-R}^{R} \frac{e^{iax}}{1+x^2}\,dx\right).$$

Let L_R be the line segment from $-R$ to R, and let $f(z) = \frac{e^{iaz}}{1+z^2}$. By the residue theorem,

$$\int_{L_R} f(z)dz + \int_{C_R} f(z)dz = 2\pi i \text{Res}(f(z); z = i) = 2\pi i \left(\frac{e^{-a}}{2i}\right) = \pi e^{-a}.$$

As

$$|\int_{C_R} f(z)dz| \leq \frac{\pi R}{R^2 - 1}$$

by part (a) and the ML inequality, we can conclude that $\lim_{R\to\infty} \int_{C_R} f(z)dz = 0$, and therefore

$$\int_{-\infty}^{\infty} \frac{\cos(ax)}{1+x^2}\, dx = \lim_{R\to\infty} \text{Re}\left(\int_{L_R} f(z)\, dz\right) = \pi e^{-a}.$$