Final Examination
January 17, 2002

There are a total of 100 possible points on this test. You will have three hours to finish the exam. You may not use any notes, books, calculators, or other materials during this exam. Please write your name on *all* blue books that you use, and be sure to write all of your answers *neatly* – illegible answers will be graded as incorrect. Good luck!!

1. (15 points)
   a. State the Cauchy-Riemann equations for the real and imaginary parts of an analytic function.
   b. Define the winding number of a closed curve γ around a point $a \notin \gamma$.
   c. State the Schwarz lemma (including the condition for equality).

2. (10 points)
   a. Determine and classify all singularities of
   $$f(z) = z^3 e^{1/z} + \frac{z^2}{(z-1)^3},$$
   and calculate the residues at each.
   b. Let $\tan(z) = \sum_{n=0}^{\infty} a_n z^n$ be the Taylor expansion of $\tan(z)$ around $z = 0$. Does the series
   $$\sum_{n=1}^{\infty} \frac{3^n a_n}{2^n}$$
   converge or diverge? Justify your answer.

3. (10 points) Let A be the complement in $\mathbf{C}$ of the non-positive real axis, i.e., $A = \mathbf{C} - \{x \in \mathbf{R} \mid x \leq 0\}$.
   a. Show that there exists a unique function f which is analytic on A and satisfies $f(x) = x^x$ for all real $x > 0$.
   b. Find $f(i)$ and $f'(i)$.

4. (10 points) Let $P(z) = a_0 + a_1 z + \cdots + a_k z^k$ be a complex polynomial. Suppose $|P(z)| \leq 1$ whenever $|z| \leq 1$. Show that $|a_n| \leq 1$ for all $n = 0, 1, \ldots, k$.

5. (15 points)

   a. State Rouché's theorem.

   b. How many zeros (counting multiplicities) does the function $f(z) = e^z - 3z^2$ have inside the unit circle?

   c. How many *distinct* zeros does f have inside the unit circle?

6. (15 points) Let $B = \{Re^{i\theta} \mid R > 0, -\pi/4 < \theta < \pi/4.\}$.

   a. Find an explicit analytic function ϕ mapping B onto the open unit disk Δ given by $|z| < 1$.

   b. Prove that there are no non-constant analytic maps $f : \mathbf{C} \to B$.

7. (10 points) Evaluate the integral

$$\int_0^{2\pi} \frac{d\theta}{3 + \cos\theta}.$$

8. (15 points) Let $R > 1$ be a real number, and let C_R be the semicircle parametrized by $z(\theta) = Re^{i\theta}$, $0 \leq \theta \leq \pi$. Let $a > 0$ be a real number.

   a. If $z \in C_R$, show that

$$\left|\frac{e^{iaz}}{1+z^2}\right| \leq \frac{1}{R^2 - 1}.$$

   b. Evaluate

$$\int_{-\infty}^{\infty} \frac{\cos(ax)}{1+x^2}\, dx.$$

   (**Hint:** Use part (a).)