Problem Set # 7 Solutions

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1. (Ch. 5, # 1) It is enough to show that f is differentiable everywhere with $f'(x) = 0$ for all x. We have

$$\left|\frac{f(t)-f(x)}{t-x}\right| \leqslant \frac{|t-x|^2}{|t-x|} = |t-x|$$

for any $t \neq x$. So,

$$f'(x) = \lim_{t\to x} \frac{f(t)-f(x)}{t-x} = 0.$$

2. (Ch. 5, # 2) Let $y = f(x)$. Then

$$g'(y) = \lim_{t\to y} \frac{g(t)-g(y)}{t-y}$$

We have $g(y) = x$. Let $s = g(t)$, so that $t = f(s)$. We claim that $s \to x$ as $t \to y$. Indeed, take any $\varepsilon > 0$ so that $[x-\varepsilon, x+\varepsilon] \subset (a, b)$. Since f is monotonic, it follows from the Intermediate Value Theorem that f maps $[x-\varepsilon, x+\varepsilon]$ bijectively onto $[f(x-\varepsilon), f(x+\varepsilon)]$. Let

$$\delta = \min(f(x) - f(x-\varepsilon), f(x+\varepsilon) - f(x)).$$

Then if $|t-y| < \delta$ then $t \in (f(x-\varepsilon), f(x+\varepsilon))$ and therefore $g(t) \in (x-\varepsilon, x+\varepsilon)$, i.e. $|s-x| < \varepsilon$. Thus,

$$g'(y) = \lim_{s\to x} \frac{s-x}{f(s)-f(x)}$$

(notice that $f(s) \neq f(x)$ for $s \neq x$ as f is injective). So,

$$g'(y) = \lim_{s\to x} \frac{1}{\frac{f(s)-f(x)}{s-x}} = \frac{1}{f'(x)}.$$

Notice that it CAN happen that $f'(x) = 0$ for a strictly increasing f (for example, $f(x) = x^3$ at $x = 0$), so for the equation in the problem to be meaningful one needs to assume that $f'(x) \neq 0$.

3. (Ch. 5, # 4) Consider the function

$$f(x) = C_0 x + \frac{C_1}{2}x^2 + \cdots + \frac{C_n}{n+1}x^{n+1}.$$

Then

$$f'(x) = C_0 + C_1 x + \cdots + C_n x^n.$$

We are given that $f(1) = 0$. It is also clear that $f(0) = 0$. By Theorem 5.9 there exist $x \in (0, 1)$ such that $f'(x) = 0$. This x is as required.

4. (Ch. 5, # 6) It is enough to show that $g'(x) > 0$. We have

$$g'(x) = \frac{xf'(x) - f(x)}{x^2},$$

so it is enough to show that $x'f(x) > f(x)$ for all $x > 0$. Applying Theorem 5.10 to f on $[0, x]$ we obtain that there exists $y \in (0, x)$ such that

$$f(x) = f(x) - f(0) = f'(y)(x - 0) = f'(y)x.$$

Since f' is monotonically increasing, we have

$$f'(x)x > f'(y)x = f(x),$$

as required.

5. (Ch. 5, # 18). We will use the Leibnitz rule:

$$(\varphi(x)\psi(x))^{(k)} = \sum_{\ell=0}^{k} \left(\begin{array}{c} k \\ \ell \end{array} \right) \varphi^{(\ell)}(x)\psi^{(k-\ell)}(x).$$

It follows that

$$f^{(k)}(t) = (t - \beta)Q^{(k)}(t) + kQ^{(k-1)}(t).$$

So,

$$f(\alpha) = f(\beta) + (\alpha - \beta)Q(\alpha),$$

and

$$f^{(k)}(\alpha) = (\alpha - \beta)Q^{(k)}(\alpha) + kQ^{(k-1)}(\alpha) \quad \text{for} \quad k = 1, \ldots, n - 1.$$

Thus,

$$P(\beta) = f(\beta) + (\alpha - \beta)Q(\alpha) + \frac{Q(\alpha) + (\alpha - \beta)Q'(\alpha)}{1!}(\beta - \alpha) + \ldots +$$

$$\frac{(\alpha - \beta)Q^{(k)}(\alpha) + kQ^{(k-1)}(\alpha)}{k!}(\beta-\alpha)^k + \frac{(\alpha - \beta)Q^{(k+1)}(\alpha) + kQ^{(k)}(\alpha)}{(k + 1)!}(\beta-\alpha)^{k+1} +$$

$$+ \ldots + \frac{(\alpha - \beta)Q^{(n-1)}(\alpha) + (n - 1)Q^{(n-2)}(\alpha)}{(n - 1)!}(\beta - \alpha)^{n-1}.$$

Notice that the terms

$$\frac{(\alpha - \beta)Q^{(k)}(\alpha)}{k!}(\beta - \alpha)^k \quad \text{and} \quad \frac{(k + 1)Q^{(k)}(\alpha)}{(k + 1)!}(\beta - \alpha)^{k+1}$$

cancel out. So, we get

$$P(\beta) = f(\beta) + \frac{(\alpha - \beta)Q^{(n-1)}(\alpha)}{(n - 1)!}(\beta - \alpha)^{n-1} =$$

$$= f(\beta) - \frac{Q^{(n-1)}(\alpha)}{(n - 1)!}(\beta - \alpha)^n,$$

and the required equation follows.