MATH 112 SET 6 SOLUTIONS

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Feel free to email waldron@fas.harvard.edu if anything is unclear. Also, if you think I misunderstood any of your arguments or was too harsh, just email me and we can meet to chat about it/negotiate (this applies for any CA).

4.1 No. Define $f(x) = 0$ if $x \neq 0$, $f(0) = 1$, the f is not continuous at 0.

Fix $x \neq 0$ and choose $0 < \delta < \epsilon < |x|$. Then $f(x + \delta) - f(x - \delta) = 0 - 0 = 0$. Furthermore, $f(0 + \delta) - f(0 - \delta) = 0 - 0 = 0$, so $\lim_{h \to 0} [f(x + h) - f(x - h)] = 0$ for all of $\mathbb{R}$ which is the desired property.

4.2 We need the statement $y \in \bar{E} \Rightarrow f(y) \in \bar{f(E)}$. Given $y \in \bar{E}$, choose a neighborhood $N_\epsilon(f(y))$; then we can choose δ such that $f(N_\delta(y)) \subset N_\epsilon(f(y))$ by continuity, and $f(N_\delta(y))$ contains points of $f(E)$ since $y \in \bar{E}$. So $f(y)$ is in $\bar{E}$.

4.8 Choose ϵ and δ as in the defn of uniform continuity. Since bounded, E can be covered by finitely many segments $\{s_n\}$ of length δ. Then $f(E)$ is covered by the finitely many segments $\{f(s_n \cap E)\}$ of diameter $< \epsilon$.

For the counterexample take the identity function on all of $\mathbb{R}$, which satisfies the conditions for continuity with $\delta := \epsilon$.

4.9 Note. A set E has $\text{diam} E < \epsilon$ iff $E \subset N_\epsilon(p)$ for any $p \in E$. Also note that $\text{diam} N_{\delta/2}(p) < \delta$.

We will show that each statement implies the other for a unif cts function f on a metric space X.

(Old $\Rightarrow$ new) Given such an ϵ, δ from the old defn of uniform continuity, choose an arbitrary nonempty set E with $\text{diam} E < \delta$; then for any $p \in E$, $E \subset N_\delta(p)$, which implies $f(E) \subset N_\epsilon(f(p))$, which (from the note) implies $\text{diam} f(E) < \epsilon$ since p was arbitrary. So δ satisfies the new defn of continuity.

(New $\Rightarrow$ old) Given ϵ, δ from the new defn of unif continuity, choose an arbitrary point $p \in X$. Then (from the note) $\text{diam} f(N_{\delta/2}(p)) < \epsilon$, which implies $f(N_{\delta/2}(p)) \subset N_\epsilon(f(p))$. So $\delta/2$ satisfies the old defn of continuity.

4.10 Note. Not uniformly continuous $\Leftrightarrow \exists\, \epsilon > 0$ s. t. for any $\delta > 0$ there exists $p \in X$ s. t. $N_\delta(p) \not\subset N_\epsilon(f(p))$, i. e. there exists p, q s. t. $d(p, q) < \delta$ and $d(f(p), f(q)) \geq \epsilon$.

Proof of theorem 4.19: Assume f is cts, but not unif cts and derive a contradiction (establishing that f is unif cts).

Given an ϵ for which f fails the defn of uniform continuity, choose points p_n, q_n as per the note, each corresponding to $\delta = 1/n$. Then $d(f(p_n), f(q_n)) \geq \epsilon$ and $d(p_n, q_n) \to 0$. Since X is compact, $\{p_n\}$ has a convergent subsequence $\{p_{m_k}\}$ converging to x, then it is clear that $\{q_{m_k}\}$ also converges to x. But then by (ordinary) continuity $f(p_{m_k}) \to f(x)$ and $f(q_{m_k}) \to f(x)$ which is impossible since $d(p_n, q_n) \geq \epsilon\ \forall\ n$.